# Using the effective graph resistance as a robustness metric

Massimo Achterberg 7 July 2022

# Outline for today

- Robustness
- Effective graph resistance
- Greedy algorithms
- Submodularity

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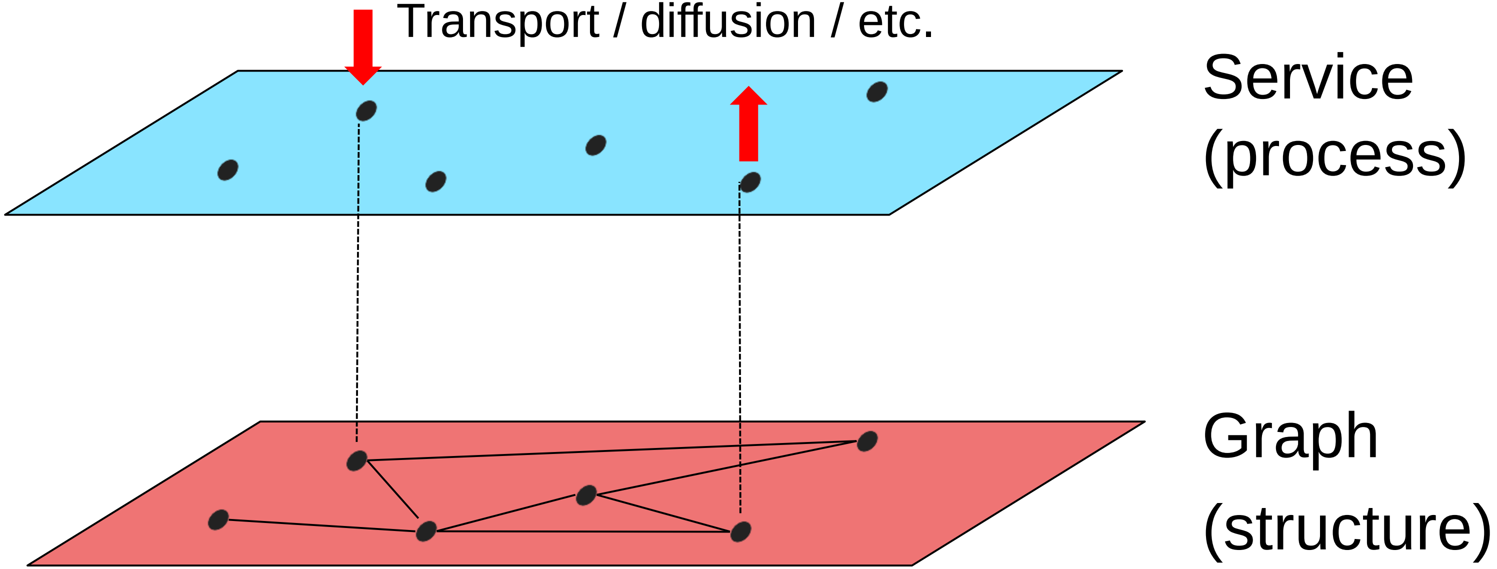
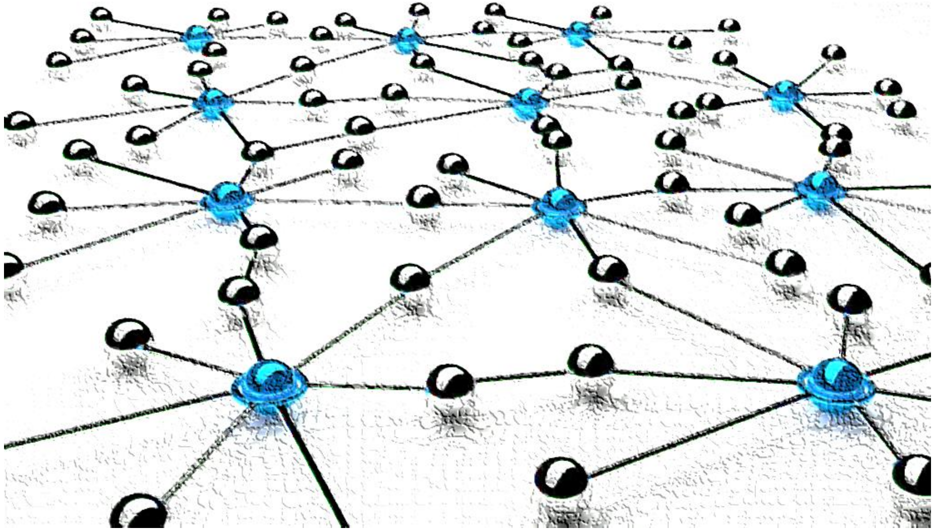
# Network = service + graph




[1] S. Trajanovski, J. Martin-Hernandez, W. Winterbach and P. Van Mieghem, 2013, "Robustness Envelopes of Networks" , Journal of Complex Networks, Vol. 1, pp. 44-62.

# What is robustness?

"The ability of a network to withstand failures and/or attacks"

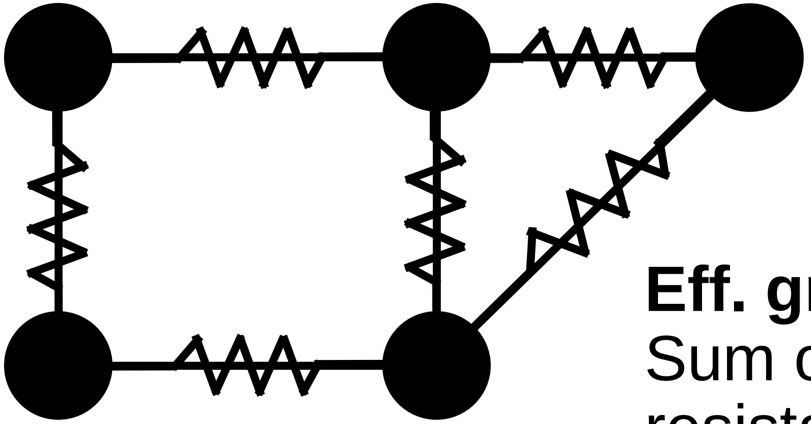


Key question: How to measure it?

# Effective graph resistance

All resistors 1Ω

Effective resistance between two nodes



Considers all paths between the nodes

**Eff. graph resistance:** Sum over all effective resistances

Other names:
resistance distance, Kirchhoff index

[2] W. Ellens, F. A. Spieksma, P. Van Mieghem, A. Jamakovic and R. E. Kooij, 2011, "Effective Graph Resistance", Linear Algebra and its Applications, Vol. 435, pp. 2491-2506.

# Given a graph G, where to add $k$ links such that the eff. graph resistance is optimal?

- Method 1: Try all possible combinations

**Number of options:** $\binom{L^c}{k} \cong \mathcal{O}\left((L^c)^k\right)$

where $L^c$ is the number of non-existent links

Infeasible!

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# Given a graph G, where to add $k$ links such that the eff. graph resistance is optimal?

- Method 2: Greedy algorithm

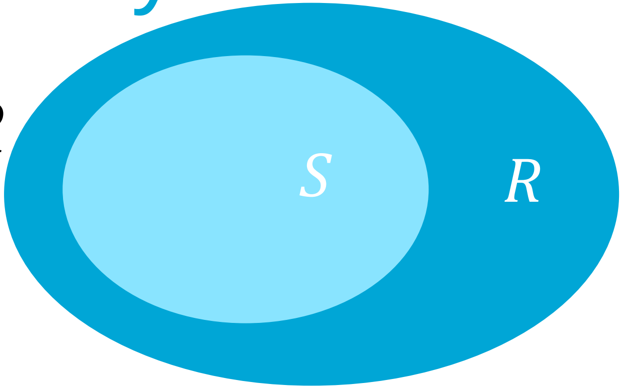
For every link $k$, check all $L^c$ options and choose the best one

**Number of options**: $\mathcal{O}(k\, L^c)$

How accurate is it?

# Definition Submodularity

- Consider two sets $S \subseteq R$
- Consider a function $f$.




- The function $f$ is **submodular** if
- Adding an element $v$ to S has a larger impact than adding v to R.

$$f(S \cup \{v\}) - f(S) \geq f(R \cup \{v\}) - f(R)$$

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[3] G.L. Newhauser, L.A. Wolsey and M.L. Fisher, 1977, "An analysis of approximations for maximizing submodular set functions - I", Mathematical Programming, Vol. 14, pp. 265-294.

# Why submodularity?

- If $f$ is sub-modular and monotone, then

$$f_{\text{greedy}} \geq \left(1 - \frac{1}{e}\right) \cdot f_{\text{optimal}}$$

$$\approx 0.63 \cdot f_{\text{optimal}}$$

Is the eff. graph resistance submodular?

No, see [4]

[4] T. Summers, I. Shames, J. Lygeros and F. Dörfler, 2017, Correction to “Topology design for optimal network coherence”, online document.

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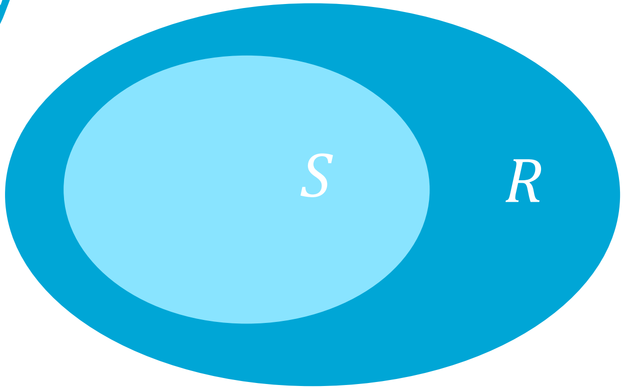
# Weak Submodularity

- Consider two sets $S \subseteq R$.
- Consider a function $f$.




- The function $f$ is weakly $\gamma$-**submodular** iff
- Adding an element $v$ to S has a larger impact than adding v to R.

$$f(S \cup \{v\}) - f(S) \geq \gamma[f(R \cup \{v\}) - f(R)],$$

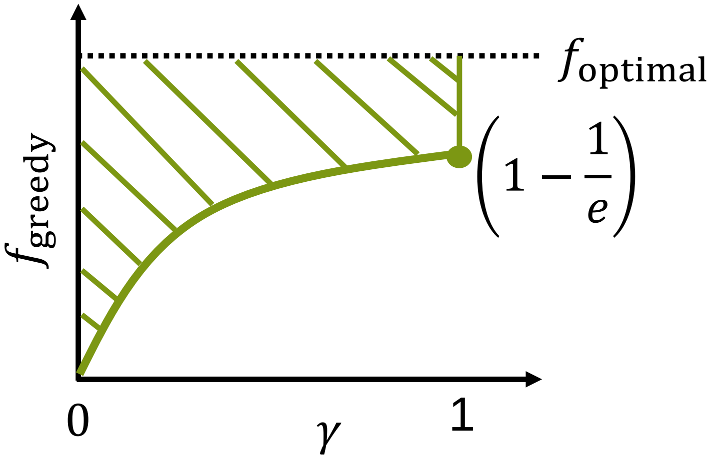
$$\gamma \in (0,1]$$

[5] T. Summers and M. Kamgarpour, "Performance guarantees for greedy maximization of non-submodular controllability metrics“, *18th European Control Conference (ECC)*, 2019, pp. 2796-2801.

# Why weak submodularity?

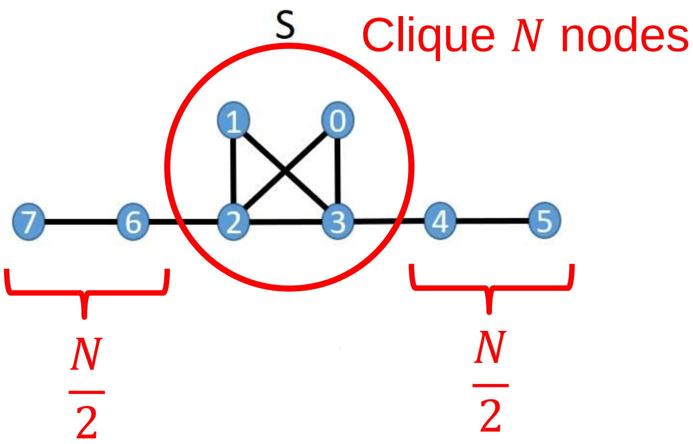
- If $f$ is weakly $\gamma$-submodular and monotone, then

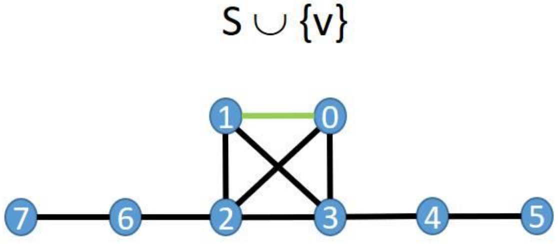
$$f_{\text{greedy}} \geq (1 - e^{-\gamma}) \cdot f_{\text{optimal}}$$



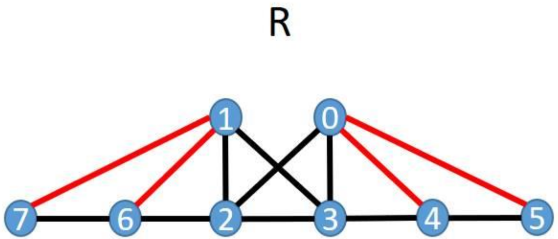

Is the eff. graph resistance weakly submodular?

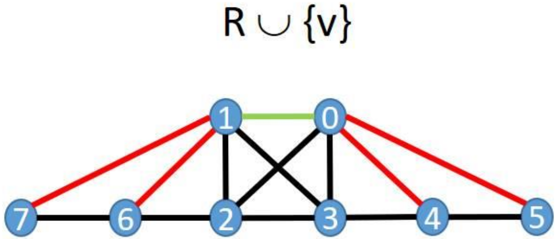
# Example




$$f(S \cup \{v\}) - f(S) = \frac{4}{N-2}$$



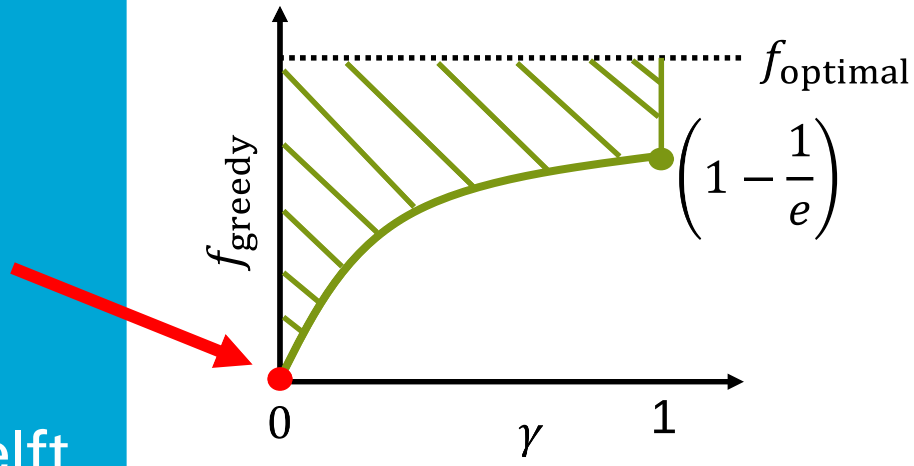




$$f(R \cup \{v\}) - f(R) \to 1$$

$$\gamma \to \frac{4}{N-2}$$

# Conclusion

- Effective graph resistance is NOT weakly $\gamma$-submodular



# Open questions – Eff. graph resistance

- Are there other algorithms with guaranteed performance?
- How well does the greedy algorithm perform in practice?
    - Seems to perform rather well
- Other types of extensions of submodularity

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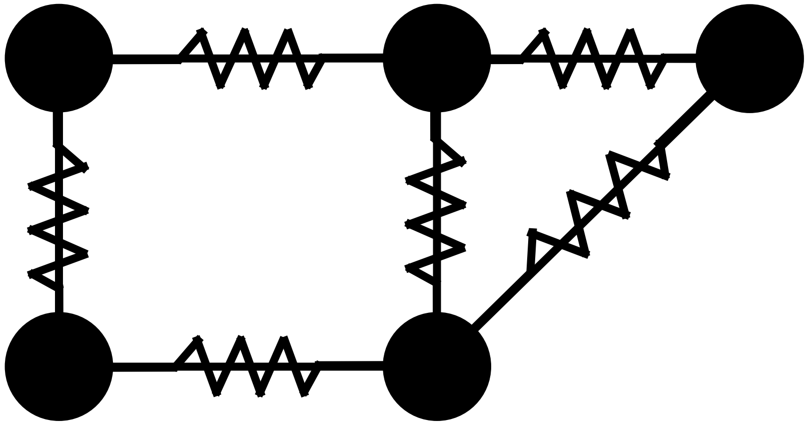
# Wrap-up

- Robustness metric: Effective graph resistance



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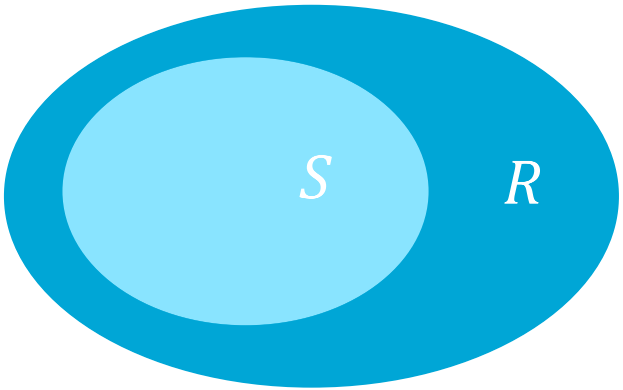
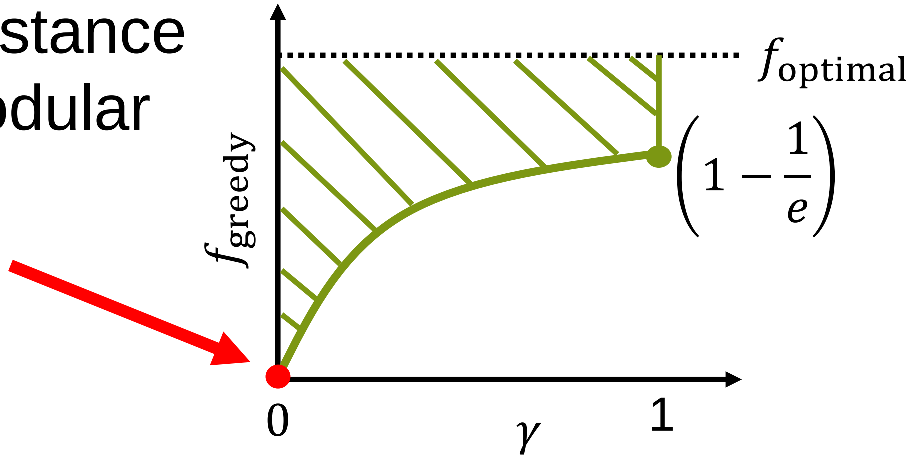
# Wrap-up




- Submodularity
- Weak submodularity

- Eff. graph resistance is NOT submodular



# Using the effective graph resistance as a robustness metric

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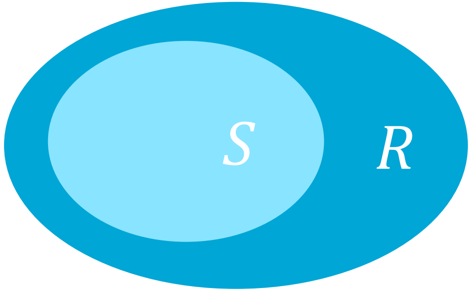
# Money ↔ Eff. Graph resistance

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$$f(S \cup \{v\}) - f(S) \geq f(R \cup \{v\}) - f(R)$$

| Item | Money | Robustness |
|---|---|---|
| Function $f$ | Measure of happiness | Eff. graph resistance |
| Set $V$ | All money in the world | All non-existing links in the graph |
| Set $S$ | A small amount of money | Subset of $V$ |
| Set $R$ | A large amount of money | Larger subset of $V$ |
| Element $v$ | Receive €1000 | Add a link |

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