

Mean Field at Distance One

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1927

Kermack & McKendrick

closed population

permanent immunity

}

\Rightarrow

incidence = $-\dot{S}$

population size = 1

incidence = ΛS

↑
force of infection

model ingredient

$A(\tau) = \underline{\text{expected}}$ contribution to the force of infection
of individual that was expected τ
units of time ago

$$\Lambda(t) = \int_0^{\infty} \text{incidence}(t-\tau) A(\tau) d\tau$$



$$\Lambda(t-\tau) s(t-\tau)$$

$$e^{-\int_{-\infty}^{t-\tau} \Lambda(\sigma) d\sigma}$$
A vertical arrow pointing upwards from the exponential term below to the source term above.

$$s(t) = e^{-\int_0^t [1 - s(t-\tau)] A(\tau) d\tau}$$

nonlinear
renewal
equation

\Downarrow

$$s(\infty) = e^{-(1-s(\infty)) R_0}$$

final size equation

$$\int_0^{\infty} A(\tau) d\tau$$

Linearization

$$1 - s(t) \approx b(t)$$

when b is small

$$b(t) = \int_0^{\infty} b(t-\tau) A(\tau) d\tau$$

Lotka's linear
renewal equation

$$R_0 = \text{expected \# of offspring} = \int_0^{\infty} A(\tau) d\tau$$

Malthusian parameter r is the real root of the
Euler-Lotka characteristic equation

$$1 = \int_0^{\infty} e^{-\lambda\tau} A(\tau) d\tau$$

$$\text{sign}(R_0 - 1) = \text{sign } r$$

$$\dot{S} = -\beta SI$$

$$\dot{I} = \beta SI - \gamma I$$

$$\dot{R} = \gamma I$$

$$S + I + R = 1$$



$$A(\tau) = \beta e^{-\gamma\tau}$$



general theory

Finite dimensional state representation of linear and nonlinear delay systems

O.D., M. Gyllenberg, J.A.J. Metz

Journal of Dynamics and Differential Equations, to appear

Heterogeneity in susceptibility

individual has n contact sites with probability π_n

proportionate
separable mixing:

probability a contact is with an individual that is of type n
equals

$$\omega_n = \frac{n\pi_n}{\sum_k k\pi_k}$$

all susceptible contact sites experience force of infection Λ

$$\dot{x} = -\Lambda x \quad , \quad x(-\infty) = 1$$

Dependence : probability an individual has n contact sites and is susceptible

$$S_n = \pi_n x^n$$

in particular

$$\dot{s}_n = -\underbrace{\Lambda n}_{\substack{\uparrow \\ \text{incoming}}} s_n$$

$$\Lambda(t) = \int_0^{\infty} \sum_n n A(\tau) \dot{s}_n(t-\tau) d\tau$$

↑
outgoing

manipulations

⋮

$$x(t) = e^{-\int_0^{\infty} A(\tau) \sum_n n \pi_n (1 - (x(t-\tau))^n) d\tau}$$

$$\Rightarrow R_0 = \sum \boxed{n^2} \pi_n \int_0^{\infty} A(\tau) d\tau$$

Networks

- repeated contacts between the same individuals
- cycles ← not incorporated

Kayin Leung, O.D.,

Dangerous connections: on binding site models of infectious disease dynamics, J. Math. Biol. (2017) 74: 619-671

O.D., M.C.M. de Jong, J.A.J. Metz

A deterministic epidemic model taking account of repeated contacts between the same individuals

J. Appl. Prob. (1998) 35: 448-462

static network with degree distribution $\{\pi_n\}$

binding site = half-edge = stub

statistically indistinguishable \iff configuration network



mean field at distance one \iff proportionate mixing

Replace $A(\tau)$ by

$F(\tau)$ = probability transmission to susceptible partner has not occurred at time τ after becoming infected

Volz variable

$x(t)$ = probability binding site has ^{at time t} a susceptible owner, conditional on the owner not becoming infected along one of its other binding sites before time t

see book of Kiss, Miller, Simon

Consistency

$$x(t) = (1) + (2)$$

(1) = probability partner is susceptible at time t

(2) = probability partner became infected some time in the past, but transmission did so far not take place

probability partner is susceptible = $g(x(t))$

$$g(x) := \sum_{n=1}^{\infty} \omega_n x^{n-1}$$

\uparrow
 $\frac{n \pi_n}{\sum_k \pi_k}$

probability per unit of time partner became infected
at time $t - \tau$

$$= -g(x(\cdot))'(t - \tau)$$

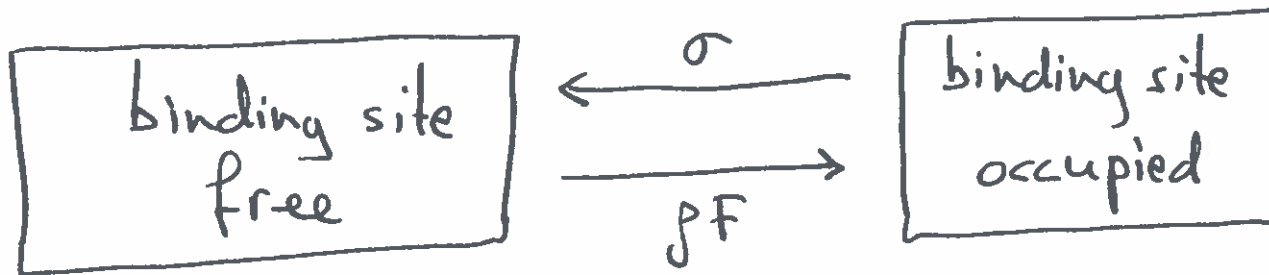
$$\Rightarrow (2) = - \int_0^{\infty} g(x(\cdot))'(t - \tau) \tilde{F}(\tau) d\tau$$

\Downarrow (by partial integration)

$$x(t) = \tilde{F}(\infty) - \int_0^{\infty} g(x(t - \tau)) \tilde{F}'(\tau) d\tau$$

$\Rightarrow R_0, r, \text{ final size}$

Dynamic configuration network



$$\frac{dF}{dt} = -\rho F^2 + \sigma(1-F)$$

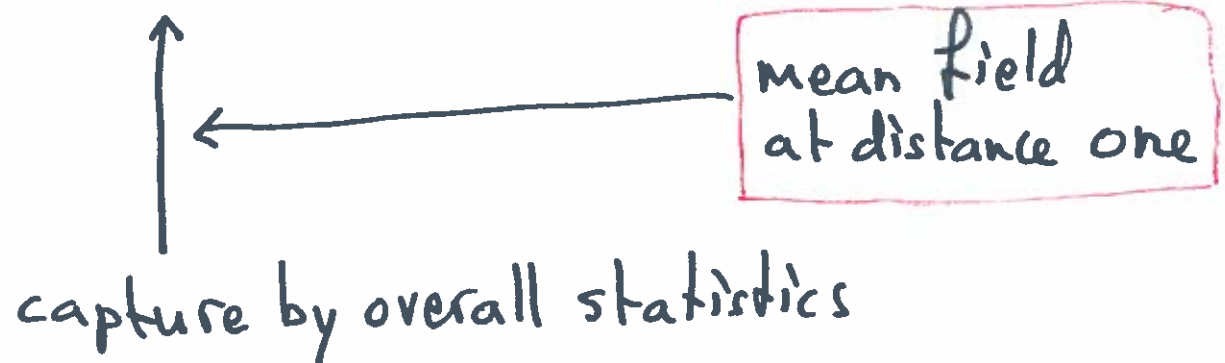
(proportionate "mixing")
(at finite positive speed)

\Downarrow
F stabilizes

bookkeeping contains information about
individuals and (the number of) their partners



but not about partners of partners



absence of degree-degree correlation

ultrafast : ✓

static : ✓ (Barbour & Reinert, Decreusefond e.a., Janson, e.a.)

dynamic without demography : ✓ (heuristics but no rigorous proof yet)

dynamic with demography : ⇔

born without partners

solution (?)
include age of partners
in bookkeeping scheme

wishful thinking

statistical description of spatio-temporal
network structure that allows for cycles
and yet is amenable for analysis of
epidemic spread in the large population limit