Metastability for the Contact Process on Finite Graphs

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Contact process

- $G = (V, E)$ locally finite, connected graph
- $\lambda > 0$ infection rate
- Contact process: Markov process $\{\xi_t\}_{t \geq 0}$ taking values on $\{0, 1\}^V$ where:

$$\xi_t(x) = 1 \to x \text{ is infected at time t}$$

$$\xi_t(x) = 0 \to x \text{ is healthy at time t}$$

- Dynamics: infected individuals recover with rate 1 and transmit with rate λ to each neighbour

Contact process

- Denote $\{\xi_t^A\}_{t\geq 0}$ the contact process with initial configuration A, that is,

$$x \in A \rightarrow \textit{infected}$$

$$x \in A^c \rightarrow \textit{healthy}$$

- "All healthy" configuration (denoted by $\underline{0}$) is absorbing
- If G is finite then $\underline{0}$ is almost surely reached

Phase transition

Definition

The extinction time of the contact process on $G = (V, E)$ *is defined by*

$$\tau_G = \inf\{t : \xi_t^V = 0\}$$

Definition

Let $G = \mathbb{Z}^d$ *and define the extinction probability as:*

$$p(\lambda) = \mathbb{P}(\exists t : \ \xi_t^{\{0\}} = \underline{0}) \text{ where } \{\xi_t^{\{0\}}\}_{t \geq 0} \text{ is C.P. with infection rate } \lambda$$

Theorem (infinite-volume phase-transition)

For each dimension d *there exists a* $\lambda_c = \lambda_c(d) \in (0, \infty)$ *such that :*

- *i)* $\lambda < \lambda_c \Rightarrow p(\lambda) = 1$ *(subcritical phase)*
- *ii)* $\lambda > \lambda_c \Rightarrow p(\lambda) < 1$ *(supercritical phase)*

Phase Transition

Definition

The box of lenght L in $\mathbb{Z}^d$ is:

$$B_L^d = \{x \in \mathbb{Z}^d : \|x\|_\infty \leq d\}$$

Theorem (finite-volume phase-transition)

For each dimension d,

- *i)* $\lambda < \lambda_c \Rightarrow \mathbb{E}(\tau_{B_L^d}) \approx C_\lambda \log(|B_L^d|)$ *(subcritical phase)*
- *ii)* $\lambda > \lambda_c \Rightarrow \mathbb{E}(\tau_{B_L^d}) \approx e^{C_\lambda |B_L^d|}$ *(supercritical phase)*

(here, $\lambda_c(d)$ is the same critical infection rate as the one for the infinite-volume phase transition)

Metastability on General Graphs

Theorem (Mountford, Mourrat, V., Yao 2016)

Let $d \geq 2$, $\lambda > \lambda_c(\mathbb{Z})$ and G be a connected graph with degrees bounded by d. Then there exists a constant $c > 0$ such that, for n large enough:

$$\mathbb{E}(\tau_G) > \exp\{c|G|\}$$

Theorem (Schapira, V. 2016)

For any $\lambda > \lambda_c(\mathbb{Z})$ and any $\epsilon > 0$, there exists a constant c_ϵ such that for any connected graph G with at least two vertices the following holds:

$$\mathbb{E}_\lambda(\tau_G) > exp\left\{c\frac{|G|}{(log(|G|))^{(1+\epsilon)}}\right\}$$

Metastability on General Graphs

Theorem (Schapira, V. 2016)

For any $\lambda > \lambda_c(\mathbb{Z})$ *and any sequence of graphs* $(G_n)_{n\in\mathbb{N}}$ *with* $|G_n| \to \infty$ *as* $n \to \infty$,

$$\frac{\tau_{G_n}}{\mathbb{E}_\lambda(\tau_{G_n})} \xrightarrow[(d.)]{n\to\infty} \exp(1)$$

Applications

- Configuration model with power law degree distribution:
  - p probability on $\mathbb{N}$ s.t. $p(\{0,1,2\}) = 0$ and $\exists c$ s.t. $p(m) \sim \frac{c}{m^a}$, $a > 2$
  - $d_1, ..., d_n \sim p$ iid half edges
  - random graph $G_n = (V_n, E_n)$, $V_n = \{1, ..., N\}$ and $deg(i) = d_i$ obtained pairing up half edges uniformly at random

Theorem (Mountford, Mourrat, V., Yao 2016)

For any $\lambda > 0$ there exists a $c > 0$ such that for all $N \in \mathbb{N}$,

$$\mathbb{E}_\lambda(\tau_{G_n}) > \exp\{cN\}$$

(stretched exponential extinction time obtained earlier by Chatterjee and Durrett)

References

Thomas Mountford, Jean-Christophe Mourrat, Daniel Valesin, Qiang Yao, *Exponential extinction time of the contact process on finite graphs*. Stochastic Processes and Their Applications (2016)

Bruno Schapira, Daniel Valesin, *Extinction time for the contact process on general graphs*. Probability Theory and Related Fields (2016)