Time Evolution of SIS epidemics in the Complete Graph

P. Van Mieghem*

Delft University of Technology
v1.: May 2013
v2: Oct 2016
v3: April 2017

Abstract

We show that, at the time being, the probability $\Pr[M(t) = k]$ that the number of infected nodes $M(t)$ at time t equals k in the Markovian continuous-time ε-SIS process on the complete graph cannot be determined exactly.

1 Introduction

In spite of the simplicity of the Markovian continuous-time SIS model, there does not seem to exist an exact time-dependent solution for any graph. Most analytic results are known for the complete graph as shown in [12, Sec. 17.6]. Before elaborating on the exact analytic solution of the Markovian continuous-time SIS model on the complete graph K_N containing N nodes, we briefly review the classical mean-field approximation.

For the complete graph K_N, mean-field approximations are accurate [3, 16]. Very likely – although there does not seem to be a rigorous proof – among all graphs, mean-field approximations are the most accurate in the complete graph. In the N-intertwined mean-field approximation (NIMFA) [15, 11], the governing equation for the probability $v(t)$ of infection in a node at time t in a regular graph G with degree r equals

$$\frac{dv(t)}{dt} = r\beta(t)\, v(t)\,(1 - v(t)) - \delta(t)\, v(t) \tag{1}$$

where the infection rate $\beta(t)$ and the curing rate $\delta(t)$ are general non-negative real functions of time t. The probability $v(t)$ at time t changes due to two possible actions: (a) if the node is healthy with probability $1 - v(t)$, its r infected neighbors – each neighbor is infected with the same probability $v(t)$ (due to symmetry) – can infect the node with instantaneous rate $\beta(t)$; (b) when the node is infected, which happens with probability $v(t)$, a curing processes with instantaneous rate $\delta(t)$ can heal the node. Since the rates are time-varying, the infection and curing process are independent, inhomogeneous Poisson processes [12]. The differential equation (1) can be solved exactly [13], resulting in

$$v(t) = \frac{\exp\left(\int_0^t (r\beta(u) - \delta(u))\, du\right)}{\frac{1}{v_0} + r\int_0^t \beta(s) \exp\left(\int_0^s (r\beta(u) - \delta(u))\, du\right) ds} \tag{2}$$

*Faculty of Electrical Engineering, Mathematics and Computer Science, P.O Box 5031, 2600 GA Delft, The Netherlands; *email*: P.F.A.VanMieghem@tudelft.nl

where v_0 is the initial fraction of infected nodes.

As shown in [6] for regular graphs, the governing differential equations are precisely the same for NIMFA and the heterogeneous mean-field (HMF) approximation [8] of Pastor-Satorras and Vespignani. Hence, the equation (1) constitutes a general SIS mean-field approximation for regular graphs. An interesting feature of (1) is its independence on the size of the network, which avoids (or ignores) finite-size effects that often complicate studies of phase transitions. For regular graphs, the NIMFA average fraction of infected nodes $y(t) = v(t)$ and $y(t)$ is coined the order parameter in statistical physics. Equation (1) with constant rates, $\beta(t) = \beta$ and $\delta(t) = \delta$, has been investigated earlier by Kephart and White [5]. Many variations on and extensions of the epidemic Kephart and White model have been proposed (see e.g. [9, 15, 7]). In fact, the differential equation (1) with constant rates has already appeared in earlier work before Kephart and White (see e.g. [2, 4]) and is also known as the logistic differential equation of population growth, first introduced by Verhulst [17] in 1845.

2 The number of infected nodes in K_N

We consider the time-dependent ε-SIS process on the complete graph, where a positive self-infection rate ε is crucial for the existence of a non-trivial steady state as shown in [12, Chapter 17]. The number of infected nodes $M(t)$ at time t in the complete graph K_N is described by a continuous-time Markov process on $\{0, 1, \ldots, N\}$ with the following rates:

$$\begin{aligned} M &\mapsto M + 1 \text{ at rate } (\beta M + \varepsilon)(N - M) \\ M &\mapsto M - 1 \text{ at rate } \delta M. \end{aligned}$$

Every infected node heals with rate δ, which explains the transition rate $M \mapsto M - 1$. Every healthy node (of which there are $N-M$ at state M) has exactly M infected neighbors, each actively transferring the virus with rate β in addition to the self-infection rate ε. Alternatively, each of the M infected nodes can infect its $N - M$ healthy neighbors with a rate β and the $N - M$ healthy nodes can infect themselves with self-infection rate ε.

This Markov process $M(t)$ is a birth and death process with birth rate $\lambda_k = (\beta k + \varepsilon)(N - k)$ and death rate $\mu_k = k\delta$ when it is in a state with $M(t) = k$ infected nodes. The steady-state probabilities $\pi_0, \ldots, \pi_N$, where $\pi_k = \lim_{t\to\infty} \Pr[M(t) = k]$, of a general birth-death process can be computed exactly [12, p. 230],[14] as

$$\pi_k = \pi_0 \binom{N}{k} \varepsilon^* \tau^{k-1} \frac{\Gamma\left(\frac{\varepsilon^*}{\tau} + k\right)}{\Gamma\left(\frac{\varepsilon^*}{\tau} + 1\right)} \qquad (k > 0) \tag{3}$$

and

$$\pi_0 = \frac{1}{\sum_{k=0}^{N} \binom{N}{k} \tau^k \frac{\Gamma\left(\frac{\varepsilon^*}{\tau} + k\right)}{\Gamma\left(\frac{\varepsilon^*}{\tau}\right)}} \tag{4}$$

where the effective infection rate $\tau = \frac{\beta}{\delta}$ and $\varepsilon^* = \frac{\varepsilon}{\delta}$. Thus, π_0 is the steady-state probability that the complete graph K_N is infection free or overall healthy. When $\varepsilon \to 0$ for N fixed, we observe from (3) that $\lim_{\varepsilon\to0} \pi_k = 0$ for $k > 0$ and, consequently, that $\lim_{\varepsilon\to0} \pi_0 = 1$, which reflects that the steady state of the SIS process (in any finite graph) is the overall-healthy state or absorbing state.

3 A generating function approach

We denote the probability that the number of infected nodes $M(t)$ at time t equals k (or that the ε-SIS process at time t is in state k) by

$$s_k(t) = \Pr[M(t) = k] \tag{5}$$

By convention, we agree that $s_k(t) = 0$ if $k > N$ or if $k < 0$. Thus, $s_0(t)$ is the probability that the epidemic dies out at time t or that the complete graph K_N is infection free at time t, but only remains infection free provided the self-infection rate $\varepsilon = 0$. Further, the steady-state probabilities

$$\pi_k = \lim_{t\to\infty} s_k(t)$$

are explicitly known in (3). The birth rate $\lambda_k = (\beta k + \varepsilon)(N - k) = -\beta k^2 + (N\beta - \varepsilon)k + N\varepsilon$ is quadratic in k and the death rate $\mu_k = \delta k$ is linear in k for any state $k \in \{0, 1, 2, \ldots, N\}$. The time-dependent evolution of the constant birth and death process [12, p. 239] as well as the linear birth and death process is described in [12, p. 243]. Here, we study the quadratic birth and death process, whose solution has, by the best of our efforts, not yet appeared in the literature.

Applying the differential equations of a general birth and death process to ε-SIS process yields the set

$$s_0'(t) = \delta s_1(t) - N\varepsilon s_0(t) \tag{6}$$

$$s_k'(t) = \left\{\beta k^2 - (N\beta + \delta - \varepsilon)k - N\varepsilon\right\} s_k(t) + \left\{-\beta(k-1)^2 + (N\beta - \varepsilon)(k-1) + N\varepsilon\right\} s_{k-1}(t) + \delta(k+1)s_{k+1}(t) \tag{7}$$

where all involved rates β, δ and ε can depend upon time t. The first differential equation (6) is incorporated in the general one (7) for $k = 0$, since $s_{-1}(t) = 0$ by our convention. If $k = N$, then $\lambda_N = 0$ as well as $s_{N+1}(t)$, so that (7) reduces to

$$s_N'(t) = -\delta N s_N(t) + \{\beta(N-1) + \varepsilon\} s_{N-1}(t)$$

Since the ε-SIS epidemic must always be in one of the possible states, there holds that $\sum_{k=0}^{N} s_k(t) = 1$.

Following the general method illustrated in [12, Sec. 11.3.3-11.3.4] for the constant and linear rate birth and death process, we start by defining the probability generating function (pgf)

$$\varphi(x,t) = E\left[x^{M(t)}\right] = \sum_{k=0}^{N} s_k(t)x^k \tag{8}$$

which we can equally well write as $\varphi(x,t) = \sum_{k=0}^{\infty} s_k(t)x^k$, according to the convention that $s_k(t) = 0$ if $k > N$ or if $k < 0$. For any probability generating function $\varphi_X(z) = E\left[z^X\right] = \sum_{k=0}^{\infty} \Pr[X = k] z^k$, the radius R of convergence around $z = 0$ in the complex z-plane is at least equal to one, because for $|z| \le 1$, it holds that $|\varphi_X(z)| \le \sum_{k=0}^{\infty} \Pr[X = k] |z|^k \le \sum_{k=0}^{\infty} \Pr[X = k] = \varphi_X(1) = 1$.

Theorem 1 *In the time-dependent ε-SIS process on the complete graph K_N, the probability generating function $\varphi(x,t)$ of the number of infected nodes $M(t)$ at time t obeys the partial differential equation*

$$\frac{\partial\varphi}{\partial t} = (x-1)\left\{-\beta x^2 \frac{\partial^2\varphi}{\partial x^2} + \{[(N-1)\beta - \varepsilon]x - \delta\}\frac{\partial\varphi}{\partial x} + N\varepsilon\varphi\right\} \tag{9}$$

Proof: After multiplying both sides in (7) by x^k and summing over all $k \geq 0$, the first line in (7) is transformed as

$$T_1 = \sum_{k=0}^{N} \left\{ \beta k^2 - (N\beta + \delta - \varepsilon) k - N\varepsilon \right\} s_k(t) x^k$$

With $\frac{\partial \varphi}{\partial x} = \sum_{k=0}^{N} k s_k(t) x^{k-1}$ and $\frac{\partial^2 \varphi}{\partial x^2} = \sum_{k=0}^{N} k (k-1) s_k(t) x^{k-2}$, we have

$$T_1 = \beta x^2 \frac{\partial^2 \varphi}{\partial x^2} - ((N-1)\beta + \delta - \varepsilon) x \frac{\partial \varphi}{\partial x} - N\varepsilon\varphi$$

Similarly, the transform of the second line in (7) taking our convention $s_{-1}(t) = 0$ into account is

$$T_2 = \sum_{k=1}^{N} \left\{ -\beta (k-1)^2 + (N\beta - \varepsilon)(k-1) + N\varepsilon \right\} s_{k-1}(t) x^k$$

leading to

$$T_2 = -\beta x^3 \frac{\partial^2 \varphi}{\partial x^2} + ((N-1)\beta - \varepsilon) x^2 \frac{\partial \varphi}{\partial x} + N\varepsilon x\varphi$$

Finally, the transform of the third and last line in (7) is, with $s_{N+1}(t) = 0$,

$$T_3 = \delta \sum_{k=0}^{N} (k+1) s_{k+1}(t) x^k = \delta \sum_{k=1}^{N+1} k s_k(t) x^{k-1} = \delta \sum_{k=1}^{N} k s_k(t) x^{k-1} = \delta \frac{\partial \varphi}{\partial x}$$

Equating the three right-hand side contributions $T_1 + T_2 + T_3$ and the transform of the left-hand side in (7) yields

$$\frac{\partial \varphi}{\partial t} = -\beta x^2 (x-1) \frac{\partial^2 \varphi}{\partial x^2} + \{(N-1)\beta x(x-1) - \varepsilon x(x-1) - \delta(x-1)\} \frac{\partial \varphi}{\partial x} + N\varepsilon (x-1)\varphi$$

Thus, we find the partial differential equation (9). □

The factor $(x-1)$ at the right-hand side of (9) is a consequence of the conservation of probability at any time t, namely that $\varphi(1,t) = \sum_{k=0}^{\infty} s_k(t) = 1$, implying that the ε-SIS stochastic process is surely in one of the possible states. Furthermore, $\left.\frac{\partial \varphi}{\partial x}\right|_{x=1} = \sum_{k=0}^{\infty} k s_k(t)$ is the average number of infected nodes at time t. Hence, the average fraction of infected nodes at time t equals

$$y(t;\tau) = \frac{1}{N} \left.\frac{\partial \varphi(x,t)}{\partial x}\right|_{x=1} \tag{10}$$

Initial condition. The ε-SIS process can start with a certain probability distribution, which then requires that the initial state vector $s(0) = (s_0(0), s_1(0), \ldots, s_N(0))$ is given. When precisely m nodes in K_N are infected initially at $t = 0$, then the boundary condition $\varphi(x,0) = \sum_{k=0}^{\infty} \delta_{km} x^k = x^m$. Clearly, the value of $m > 0$ must exceed zero, because $\varphi(0,t) = s_0(t)$ is the probability that the complete graph is infection free at time t and, on the long run, $\lim_{t\to\infty} \varphi(0,t) = \pi_0$ is given by (4).

Confinement. *In the sequel, we limit ourselves to constant rates: none of the infection rate β, self-infection rate ε or curing rate δ is a function of time t. In addition, we assume that the ε-SIS process starts at $t = 0$.*

3.1 The steady-state probability generating function $\varphi_\infty(x)$

The steady-state probability generating function (assuming constant rates) equals with (3)

$$\lim_{t\to\infty}\varphi(x,t)=\sum_{k=0}^{\infty}\lim_{t\to\infty}\Pr\left[M\left(t\right)=k\right]x^k=\sum_{k=0}^{\infty}\pi_k x^k=\varphi_\infty(x)$$

where

$$\varphi_\infty(x)=\pi_0+\frac{\varepsilon^*\pi_0}{\tau\Gamma\left(\frac{\varepsilon^*}{\tau}+1\right)}\sum_{k=1}^{N}\binom{N}{k}\Gamma\left(\frac{\varepsilon^*}{\tau}+k\right)\left(\tau x\right)^k \tag{11}$$

Thus, if $\varepsilon=0$, then $\pi_0=1$ and there holds that $\lim_{t\to\infty}\varphi(x,t)=\varphi_\infty(x)=1$. If $\varepsilon>0$, the steady-state probability generating function $\varphi_\infty(x)$ is a polynomial of degree N in x, which is more elegantly written as

$$\varphi_\infty(x)=\frac{\pi_0}{\Gamma\left(\frac{\varepsilon^*}{\tau}\right)}\sum_{k=0}^{N}\binom{N}{k}\Gamma\left(\frac{\varepsilon^*}{\tau}+k\right)\left(\tau x\right)^k \tag{12}$$

and the general relation for any pgf, $\varphi_\infty(1)=1$, also follows from (4). Finally, $\varphi_\infty(x)$ is a function of three parameters

$$\varphi_\infty(x)=\varphi_\infty(x;\tau,\varepsilon^*,N)$$

The partial differential equation (9) simplifies, in the steady state for $t\to\infty$ and $\frac{\partial\varphi}{\partial t}=0$, to

$$-\beta x^2\frac{\partial^2\varphi_\infty}{\partial x^2}+\left\{\left[\left(N-1\right)\beta-\varepsilon\right]x-\delta\right\}\frac{\partial\varphi_\infty}{\partial x}+N\varepsilon\varphi_\infty=0 \tag{13}$$

Introducing the integral for the Gamma function $\Gamma\left(s\right)=\int_0^\infty u^{s-1}e^{-u}du$, valid for $\operatorname{Re}\left(s\right)>0$, into (12) yields

$$\begin{aligned}\varphi_\infty(x)&=\frac{\pi_0}{\Gamma\left(\frac{\varepsilon^*}{\tau}\right)}\sum_{k=0}^{N}\binom{N}{k}\left(\tau x\right)^k\int_0^\infty u^{\frac{\varepsilon^*}{\tau}+k-1}e^{-u}du\\&=\frac{\pi_0}{\Gamma\left(\frac{\varepsilon^*}{\tau}\right)}\int_0^\infty u^{\frac{\varepsilon^*}{\tau}-1}e^{-u}\left\{\sum_{k=0}^{N}\binom{N}{k}\left(u\tau x\right)^k\right\}du\end{aligned}$$

Invoking Newton's binomium leads to an integral representation[1] of the steady-state probability generating function for $\varepsilon>0$,

$$\varphi_\infty(x;\tau,\varepsilon^*,N)=\frac{\pi_0}{\Gamma\left(\frac{\varepsilon^*}{\tau}\right)}\int_0^\infty u^{\frac{\varepsilon^*}{\tau}-1}e^{-u}\left(1+u\tau x\right)^N du \tag{15}$$

[1] Assuming a positive real x and letting $w=\left(\tau x\right)u$, we find

$$\varphi_\infty(x)=\frac{\pi_0}{\left(\tau x\right)^{\frac{\varepsilon^*}{\tau}}\Gamma\left(\frac{\varepsilon^*}{\tau}\right)}\int_0^\infty e^{-\frac{w}{\tau x}}w^{\frac{\varepsilon^*}{\tau}-1}\left(1+w\right)^N dw$$

We conclude that the steady-state probability generating function $\varphi_\infty(x)$ can be written as

$$\varphi_\infty(x)=\frac{\pi_0}{\left(\tau x\right)^{\frac{\varepsilon^*}{\tau}}}U\left(\frac{\varepsilon^*}{\tau},\frac{\varepsilon^*}{\tau}+1+N,\frac{1}{\tau x}\right) \tag{14}$$

where the confluent hypergeometric function [1, 13.2.8]

$$U\left(a,b,z\right)=\frac{1}{\Gamma\left(a\right)}\int_0^\infty e^{-zt}w^{a-1}\left(1+w\right)^{b-a-1}dt$$

is one of the independent solutions of Kummer's differential equation $x\frac{d^2f}{dx^2}+\left(b-x\right)\frac{df}{dx}-af=0$ (see e.g. [1, Chapter 13]).

3.2 General solution of the partial differential equation (9)

Theorem 2 *In the time-dependent ε-SIS process on the complete graph K_N with constant infection rate β, self-infection rate ε and curing rate δ, the probability generating function $\varphi(x,t)$ of the number of infected nodes $M(t)$ at time t can be written as a Laplace transform*

$$\varphi(x,t) = \int_0^\infty e^{-ct} g(x;c)\, dc \tag{16}$$

where the function $g(x,c)$ obeys the differential equation

$$-x^2(x-1)\frac{d^2g}{dx^2} + \left\{\left[(N-1) - \frac{\varepsilon^*}{\tau}\right]x - \frac{1}{\tau}\right\}(x-1)\frac{dg}{dx} + \frac{1}{\tau}\left(N\varepsilon^*(x-1) + c^*\right)g = 0 \tag{17}$$

and $\tau = \frac{\beta}{\delta}$, $\varepsilon^* = \frac{\varepsilon}{\delta}$ and $c^* = \frac{c}{\delta} \geq 0$.

Proof: The usual recipe of the separation of the variables t and x, by assuming that a solution in product form as $\varphi(x,t) = g(x)\,h(t)$ exists, transforms (9) to

$$\frac{\partial \log h}{\partial t} = \frac{(x-1)}{g}\left\{-\beta x^2 \frac{d^2 g}{dx^2} + \{[(N-1)\beta - \varepsilon]x - \delta\}\frac{dg}{dx} + N\varepsilon g\right\} \tag{18}$$

By taking the derivative of both sides with respect to x, we find with $\frac{\partial}{\partial x}\frac{\partial \log h}{\partial t} = 0$ that

$$\frac{(x-1)}{g}\left\{-\beta x^2 \frac{d^2 g}{dx^2} + \{[(N-1)\beta - \varepsilon]x - \delta\}\frac{dg}{dx} + N\varepsilon g\right\} = c_1 \tag{19}$$

where c_1 is a constant that is neither a function of x nor of t, because the left-hand side in (19) is independent of t. Similarly, by taking the derivative of both sides in (18) with respect to t, we find that

$$\frac{\partial \log h}{\partial t} = c_2 \tag{20}$$

and (18) shows that $c_1 = c_2 = -c$.

We rewrite (19) with $\varepsilon^* = \frac{\varepsilon}{\delta}$ and $c^* = \frac{c}{\delta}$ to find (17).

From (20), we find $h(t) = h(0)\,e^{-ct}$ for the time $t \geq 0$. If c were complex and $\mathrm{Im}(c) \neq 0$, then $h(t) = h(0)\,e^{-\mathrm{Re}(c)t}(\cos t\,\mathrm{Im}(c) + i \sin t\,\mathrm{Im}(c))$ and $\varphi(x,t) = g(x)\,h(t)$ is generally complex for $t > 0$. However, the definition (8) of the pgf $\varphi(x,t)$ illustrates that $\varphi(x,t)$ is real for real x at any time $t \geq 0$. Hence, c must be real. Moreover, since the asymptotic pgf $\lim_{t\to\infty}\varphi(x,t) = \varphi_\infty(x)$ exists, c must be non-negative, otherwise $\lim_{t\to\infty} h(t) = h(0)\lim_{t\to\infty} e^{-ct} = \infty$. We conclude that the eigenvalue c is real and non-negative.

The general solution of the eigenvalue differential equation in c consists of a linear combination $\sum_{c\geq 0} e^{-ct} g(x;c)$ if the eigenvalues c are discrete. Generally, one readily verifies that $\varphi(x,t) = \int_0^\infty e^{-ct} g(x;c)\, dc$ satisfies the partial differential equation (9) provided that $g(x;c)$ is a solution of the differential equation (17) as a function of the "eigenvalue" c. □

In fact, we need to solve an eigenvalue problem that can be expanded in a Sturm-Liouville series [10]. For $c = 0$, the differential equation (17) reduces to the differential (13) and we conclude that

$$g(x,0) = \varphi_\infty(x)$$

The ε-SIS process on the complete graph K_N with N nodes is described by a general birth-death process by the differential equations (6) and (7). This set of linear differential equations possesses a general $(N+1) \times (N+1)$ tri-diagonal matrix, whose eigenstructure is studied in depth in [12, A.6.3]. The $N+1$ non-negative, real eigenvalues (and one of them is zero) imply that the eigenvalues c are a discrete set $\{c_0 = 0, c_1, \ldots, c_N\}$, so that the Laplace integral in (16) will reduce to a sum $\varphi(x,t) = \varphi_\infty(x) + \sum_{k=1}^{N} e^{-c_k t} g(x; c_k)$ for finite size N.

The second-order differential equation (17) in the function g is of the type

$$x^2 (1-x) \frac{d^2 g}{dx^2} + (ax+b)(1-x) \frac{dg}{dx} + (\lambda + d(1-x)) g = 0 \tag{21}$$

where $a = \frac{\varepsilon^*}{\tau} - (N-1)$, $b = \frac{1}{\tau}$, $d = N\frac{\varepsilon^*}{\tau}$ and $\lambda = \frac{c^*}{\tau}$ are real numbers. Unfortunately, (21) does not seem to be of a known type. Gauss's hypergeometric function $F(a,b;c;x)$ obeys the differential equation [1, Chapter 15]

$$x(1-x) \frac{d^2 g}{dx^2} + [c - (a+b+1)x] \frac{dg}{dx} - abg = 0$$

Slightly more general, (17) is of the type

$$p_3(x) g^{(2)}(x) + p_2(x) g^{(1)}(x) + p_1(x) g(x) = 0$$

where $p_k(x)$ is a polynomial in x of degree k, where the hypergeometric differential equation is of the form

$$p_2(x) g^{(2)}(x) + p_1(x) g^{(1)}(x) + p_0(x) g(x) = 0$$

In conclusion, unless an analytic solution of the differential (21) can be found, we are afraid that the probability $s_k(t) = \Pr[M(t) = k]$, that the number of infected nodes $M(t)$ at time t equals k in the Markovian continuous-time ε-SIS process on the complete graph K_N, cannot be determined exactly.

Acknowledgement. I am grateful to Johan Dubbeldam for checking the computations and for useful discussions.

References

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions.* Dover Publications, Inc., New York, 1968.

[2] N. T. J. Bailey. *The Mathematical Theory of Infectious Diseases and its Applications.* Charlin Griffin & Company, London, 2nd edition, 1975.

[3] E. Cator and P. Van Mieghem. Susceptible-Infected-Susceptible epidemics on the complete graph and the star graph: Exact analysis. *Physical Review E*, 87(1):012811, January 2013.

[4] D. J. Daley and J. Gani. *Epidemic modelling: An Introduction.* Cambridge University Press, Cambridge, U.K., 1999.

[5] J. O. Kephart and S. R. White. Direct-graph epidemiological models of computer viruses. *Proceedings of the 1991 IEEE Computer Society Symposium on Research in Security and Privacy*, pages 343–359, May 1991.

[6] C. Li, R. van de Bovenkamp, and P. Van Mieghem. Susceptible-infected-susceptible model: A comparison of N-intertwined and heterogeneous mean-field approximations. *Physical Review E*, 86(2):026116, August 2012.

[7] R. Pastor-Satorras, C. Castellano, P. Van Mieghem, and A. Vespignani. Epidemic processes in complex networks. *Review of Modern Physics*, 87(3):925–979, September 2015.

[8] R. Pastor-Satorras and A. Vespignani. Epidemic dynamics and endemic states in complex networks. *Physical Review E*, 63:066117, 2001.

[9] R. Pastor-Satorras and A. Vespignani. Epidemic spreading in scale-free networks. *Physical Review Letters*, 86(14):3200–3203, April 2001.

[10] E. C. Titchmarsh. *Eigenfunction Expansions Associated with Second-order Differential Equations, Part I.* Oxford University Press, Amen House, London, 2nd edition, 1962.

[11] P. Van Mieghem. The N - Intertwined SIS epidemic network model. *Computing*, 93(2):147–169, 2011.

[12] P. Van Mieghem. *Performance Analysis of Complex Networks and Systems.* Cambridge University Press, Cambridge, U.K., 2014.

[13] P. Van Mieghem. SIS epidemics with time-dependent rates describing ageing of information spread and mutation of pathogens. Delft University of Technology, Report20140615 (www.nas.ewi.tudelft.nl/people/Piet/TUDelftReports), 2014.

[14] P. Van Mieghem and E. Cator. Epidemics in networks with nodal self-infections and the epidemic threshold. *Physical Review E*, 86(1):016116, July 2012.

[15] P. Van Mieghem, J. Omic, and R. E. Kooij. Virus spread in networks. *IEEE/ACM Transactions on Networking*, 17(1):1–14, February 2009.

[16] P. Van Mieghem and R. van de Bovenkamp. Accuracy criterion for the mean-field approximation in SIS epidemics on networks. *Physical Review E*, 91(3):032812, March 2015.

[17] P. F. Verhulst. Recherches mathématiques sur la loi d'accroissement de la population. *Nouveaux Mémoires de l'Académie Royale des Sciences et des Belles-Lettres de Bruxelles*, XVIII:1–41, 1845.

A Reduction of the differential equation (17) to the standard form

We aim to transform (21) into the form [10]

$$\frac{d^2y}{du^2} + (\lambda - q(u))\, y(u) = 0 \tag{22}$$

The standard form has many interesting properties. First, the Wronskian is constant in u. Second, Titchmarsh [10] gives, at the beginning of the chapters, insight in the spectrum of λ and he also presents bounds to the solution y.

We make the transformation $x = h(u)$, so that $u = h^{-1}(x)$. Thus, by using the chain rule and denoting $f(u) = g(h(u))$, we have

$$\frac{dg(x)}{dx} = \frac{d}{du} g(h(u)) \frac{du}{dx} = \frac{df}{du} \frac{1}{\frac{dx}{du}} = \frac{1}{h'(u)} \frac{df}{du}$$

and

$$\frac{d^2g(x)}{dx^2} = \frac{d}{dx}\left(\frac{dg(x)}{dx}\right) = \frac{d}{du}\left(\frac{dg(x)}{dx}\right)\frac{du}{dx} = \frac{1}{(h'(u))^2}\frac{d^2f}{du^2} - \frac{h''(u)}{(h'(u))^3}\frac{df}{du}$$

We obtain

$$\frac{h^2(1-h)}{(h'(u))^2}\frac{d^2f}{du^2} + \frac{(1-h)}{h'(u)}\left\{(ah+b) - h^2\frac{h''(u)}{(h'(u))^2}\right\}\frac{df}{du} + (\lambda + d(1-h))\, f = 0 \tag{23}$$

Nex, we choose h such that $h^2(1-h)\frac{1}{(h'(u))^2} = 1$. Thus, $h^2(1-h) = (h'(u))^2$ or $\frac{dh}{du} = \pm h\sqrt{1-h}$ and integrated

$$\pm\int \frac{dh}{h\sqrt{1-h}} = u$$

As in Tichmarsh [10], we assume the positive sign and find

$$u = \log\frac{1-\sqrt{1-h}}{1+\sqrt{1-h}}$$

and, inversed,

$$x = h(u) = \operatorname{sech}^2\left(\frac{u}{2}\right)$$

Thus, $x = h(u) = \operatorname{sech}^2\left(\frac{u}{2}\right)$ and $u = 2\text{ArcSech}(\sqrt{x})$, which is only real and positive for $x \in (0,1)$.

After introducing $h(u) = \operatorname{sech}^2\left(\frac{u}{2}\right)$ into (23) yields

$$\frac{d^2f}{du^2} + \left\{\frac{\cosh u - 2}{\sinh u} - a\tanh\frac{u}{2} - \frac{b}{2}\sinh u\right\}\frac{df}{du} + \left(\lambda + d\tanh^2\frac{u}{2}\right) f = 0$$

Let

$$r(u) = \frac{\cosh u - 2}{\sinh u} - a\tanh\frac{u}{2} - \frac{b}{2}\sinh u \tag{24}$$

then we obtain the differential equation in $f(u) = g\left(\operatorname{sech}^2\left(\frac{u}{2}\right)\right)$ and $x = \operatorname{sech}^2\left(\frac{u}{2}\right)$ or $u = 2\text{ArcSech}(\sqrt{x})$,

$$\frac{d^2f}{du^2} + r(u)\frac{df}{du} + \left(\lambda + d\tanh^2\frac{u}{2}\right) f = 0$$

We proceed with the reduction to the standard form by considering $f(u) = p(u)\, s(u)$ and the above differential equation becomes

$$0 = p''(u) + p'(u)\left\{2\frac{s'(u)}{s(u)} + r(u)\right\} + p(u)\left\{\frac{s''(u)}{s(u)} + r(u)\frac{s'(u)}{s(u)} + \left(\lambda + d\tanh^2\frac{u}{2}\right)\right\}$$

The standard form requires that $2\frac{s'(u)}{s(u)} + r(u) = 0$, or

$$2\frac{d}{du}\log s(u) = -r(u)$$

and

$$s(u) = \exp\left(-\frac{1}{2}\int r(u)\,du\right)$$

Explicitly, we have

$$s(u) = \exp\left(-\frac{1}{2}\int r(u)\,du\right) = \frac{\tanh u\left(\cosh\frac{u}{2}\right)^a}{\sqrt{\sinh u}}e^{\frac{b}{4}\cosh u} \tag{25}$$

From $2\frac{s'(u)}{s(u)} + r(u) = 0$ or $\frac{s'(u)}{s(u)} = -\frac{1}{2}r(u)$, equivalent to $2s'(u) + s(u)\,r(u) = 0$, we find that

$$s''(u) = -\frac{1}{2}s'(u)\,r(u) - \frac{1}{2}s(u)\,r'(u)$$

which we use in

$$X = \frac{s''(u)}{s(u)} + r(u)\frac{s'(u)}{s(u)} = -\frac{1}{4}\left\{r^2(u) + 2r'(u)\right\}$$

Hence, with $s(u)$ in (25) and obeying $\frac{s'(u)}{s(u)} = -\frac{1}{2}r(u)$ and with $p(u) = \frac{f(u)}{s(u)}$, we arrive at

$$p''(u) + p(u)\left\{\lambda + d\tanh^2\frac{u}{2} - \frac{1}{4}\left\{r^2(u) + 2r'(u)\right\}\right\} = 0$$

so that

$$q(u) = \frac{1}{4}\left\{r^2(u) + 2r'(u)\right\} - d\tanh^2\frac{u}{2} \tag{26}$$

We now compute $q(u)$. From the definition (24) of $r(u)$,

$$\begin{aligned}r^2(u) + 2r'(u) = {} & 1 - a + a(a+1)\tanh^2\frac{u}{2} + b\left\{\frac{b}{4}\cosh u + \frac{b}{4} - 2\right\}(\cosh u - 1) + \frac{1}{\sinh^2 u}\\ & + a\tanh\frac{u}{2}\left(b\sinh u - 2\frac{\cosh u - 2}{\sinh u}\right)\end{aligned}$$

which is not such an insightfull expression!

Finally, we arrive with $p(u) = \frac{f(u)}{s(u)}$ at the standard form

$$p''(u) + \left(\lambda + d\tanh^2\frac{u}{2} - \frac{1}{4}\left\{r^2(u) + 2r'(u)\right\}\right)p(u) = 0$$

Explicitly, with $\lambda' = \lambda + \frac{a}{4} - \frac{b}{2}$

$$p''(u) + \left(\begin{array}{c}\lambda' + \left(d - \frac{a^2}{4} - \frac{a}{4}\right)\tanh^2\frac{u}{2} - \frac{b^2}{16}\sinh^2 u - \frac{\cosh^2 u + 2}{4\sinh^2 u} - \frac{ab}{4}\sinh u\tanh\frac{u}{2}\\ + \frac{1}{2}a\tanh\frac{u}{2}\frac{\cosh u - 2}{\sinh u} + \frac{b}{2}\cosh u\end{array}\right)p(u) = 0$$