New lower bounds for the fundamental weight of the principal eigenvector in complex networks

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***Abstract*—The principal eigenvector x_1 belonging to the largest adjacency eigenvalue (*i.e.* the spectral radius) λ_1 of a graph is one of the most popular centrality metrics. The spectral radius λ_1 of the adjacency matrix powerfully characterizes the dynamic processes on networks, such as virus spread and synchronization. The sum of components of the principal eigenvector, which is also called the fundamental weight w_1, is argued to be as important as the eigenvalues of the graph matrix. Here we theoretically prove two new types of lower bound w_L and w_D for the fundamental weight w_1 in any network. The lower bound w_L is related to the clique number (the size of the largest clique) of the network. The lower bound w_L is sharper than the w_D whereas the computational complexity of w_D is lower. We compare the sharper lower bound w_L with w_1 in different networks. The effect of the network structure on the relative deviation of w_L is studied. Based on w_L, another new lower bound for w_1 is proposed for a special type of networks.**

I. INTRODUCTION

The largest eigenvalue λ_1 of the adjacency matrix A, called the spectral radius of the graph, has been shown to play an important role in dynamic processes on graphs, such as SIS (susceptible-infected-susceptible) virus spread [1], [2], [3] on a given network topology. In the past decade, researches have focused on how topological changes, such as link (or node) removal, may alter the spectral radius. Van Mieghem *et al.* [4] studied link removal strategies that minimize the spectral radius and showed that the best strategy to minimize the spectral radius is based on the components of the principal eigenvector x_1. It underlines the importance to understand x_1. Moreover, in susceptible-infected-susceptible (SIS) epidemic processes, the meta-stable state infection vector $V_\infty = \varsigma x_1$ when the effective spreading rate $\tau = \tau_c^{(1)} + \varsigma$, where $\tau_c^{(1)} = 1/\lambda_1$ is the lower bound of the exact SIS epidemic threshold and $\varsigma > 0$ is an arbitrary small constant [5]. In other words, x_1 is proportional to the infection probabilities of the nodes in the meta-stable state with an effective spreading rate that is just above the epidemic threshold. In this case, the fundamental weight $w_1 = u^T x_1$ where u is the all-one vector, is thus proportional to the number of infected nodes.

Furthermore, nodal centrality metrics quantify the "importance" of a node in a network or how "central" a node is in the graph. Many quantifiers of nodal "importance" have been proposed in literature [6], [7], [8], [9], [10]. The principal eigenvector of complex networks is one of the most popular nodal centrality metrics [11], [12], [13]. Li *et al.* [14], [15] have studied the influence of the assortativity[1] on the principal eigenvector and the relation between the principal eigenvector and other centrality metrics.

However, there is currently no better lower bound for the fundamental weight w_1 of the principal eigenvector than 1 (see [17]). In this work, we propose some new lower bounds for w_1 and study how sharp the lower bounds are in interconnected networks. In this work, we consider the interconnected networks that are composed of a clique and a random network, that are randomly interconnected. The choice of such interconnected networks is motivated by: (1) the fact that most real-world complex networks are not isolated but instead interconnected. These interconnected networks are interdependent and present different structural and dynamical features from those observed in isolated networks [18], [19], [20], [21], [22], [23]; (2) many social networks can be modeled as a clique randomly interconnected with a random network. For example, the club organizers or the company leaders are completely connected to each other forming a clique, while other non-critical persons are randomly connected to each other and to the clique; (3) the size of the largest clique of such interconnected networks could be approximately controlled in our interconnected network model. It offers a possibility to study the influence of the size of the largest clique on how tight the lower bounds for w_1 are.

This paper is organized as follows. In Section II we derive two new lower bounds w_L and w_D (see Eqs. (1) and (4)), which are related to the size of the largest clique and the largest degree, for the fundamental weight w_1 in any network. In Section III we compare w_1 and w_L in interconnected networks with different topological features. In Sec. IV, we study the influence of the number and the location of the interconnections on the difference between w_L and w_1. In Sec. V, we propose another new lower bound w_s for the interconnected networks introduced in Sec. III-A. Finally, we conclude in Sec. VI.

II. NEW LOWER BOUNDS FOR THE FUNDAMENTAL WEIGHT OF THE PRINCIPAL EIGENVECTOR

We consider a network $G(\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{L}$ is the set of links. The number of nodes is denoted by $N = |\mathcal{N}|$ and the number of links by $L = |\mathcal{L}|$. The network

[1] Assortativity ρ_D is also called the degree correlation, is computed as the linear correlation coefficient of the degree of nodes connected by a link. The assortativity describes the tendency of network nodes to connect preferentially to other nodes with either similar (when $\rho_D > 0$) or opposite (when $\rho_D < 0$) degree [16].

G can be represented by an $N \times N$ symmetric adjacency matrix A, consisting of elements a_{ij}, which are either one or zero depending on whether node i is connected to node j or not. The networks mentioned in this paper are simple, unweighted without self-loops nor multiple links. The largest eigenvalue λ_1 of the adjacency matrix A is also called the spectral radius [17]. The principal eigenvector x_1 corresponding to the spectral radius λ_1 satisfies the eigenvalue equation

$$Ax_1 = \lambda_1 x_1.$$

The j-th component of the principal eigenvector is denoted by $(x_1)_j$. We call $w_1 = \sum_{i \in \mathcal{N}} (x_1)_i$ the fundamental weight of the principal eigenvector [13].

The size of cliques in G is denoted as $\omega_1, \omega_2, \cdots, \omega_n$, where n is the number of cliques in a network, which we order as $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$. The size ω_1 of the largest clique is called the clique number of G.

It is known [17] that the fundamental weight w_1 is upper bounded by $w_1 \leq \sqrt{N}$ in any network, and the equality occurs for regular graphs, *i.e.* all degrees are equal. Here we give two new lower bounds for w_1.

Theorem 1: In any network, the fundamental weight of w_1 is lower bounded by

$$w_1 \geq w_L = \sqrt{\frac{\lambda_1}{1 - 1/\omega_1}} \tag{1}$$

Proof: The Motzkin-Straus theorem [24], [25] asserts that

$$1 - \frac{1}{\omega_1} = \max_{x \in S} x^T Ax, \tag{2}$$

where the simplex S contains all vectors x that lie in the hyperplane $u^T x = 1$ (*i.e.* u is the all-one vector) and possess non-negative components. For vectors x normalized as $x^T x = 1$, the Rayleigh inequalities demonstrate that $x^T Ax \leq \lambda_1$, with equality only if $x = x_1$ is the (normalized) eigenvector of A belonging to the spectral radius λ_1. When choosing $x = \frac{x_1}{u^T x_1}$ in (2), Wilf [25] found that

$$(1 - \frac{1}{\omega_1}) = \max_{x \in S} x^T Ax \geq \frac{x_1^T Ax_1}{(u^T x_1)^2} = \frac{\lambda_1}{w_1^2}, \tag{3}$$

where $w_1 \geq 1$ (see [13]). Wilf's bound leads to the lower bound (1) for the fundamental weight w_1. ■

Theorem 2: In any network, the fundamental weight of w_1 is lower bounded by

$$w_1 \geq w_D = \sqrt{\frac{\lambda_1}{1 - 1/d_{\max}}} \tag{4}$$

Proof: In any network, the clique number ω_1 is not larger than the largest degree $d_{\max}$. With Eq. (1) and $\omega_1 \leq d_{\max}$, Eq. (4) is proved. ■

Finding the clique number ω_1 of a graph is an NP-hard problem [26], [27]. Hence, the computational complexity of the lower bound w_D is far lower than that of w_L, although $w_L \geq w_D$, *i.e.* w_L is a tighter lower bound than w_D. We compare the fundamental weight w_1 and its lower bound w_D in ER networks with different densities (see Fig. 1). We find that the relative deviation $\frac{w_1 - w_D}{w_1}$ decreases with the increase of the link density $p = L/\binom{N}{2}$ of networks. It means that w_D is a good lower bound for w_1 in networks with a large link density.

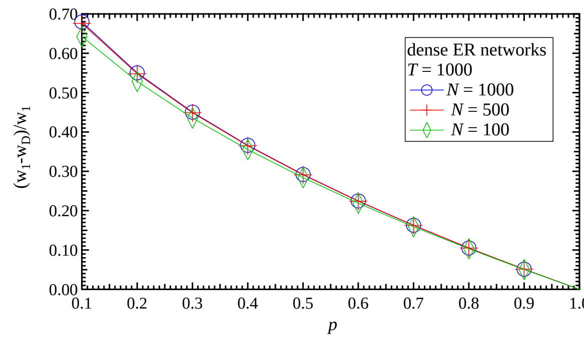

Fig. 1: *Relative deviation $\frac{w_1 - w_D}{w_1}$ as a function of the link density p in ER networks ($N = 1000$). The simulations are performed on 10^3 realizations.*

We mainly focus on the tighter lower bound w_L in the rest of this paper.

III. Comparison of the Fundamental Weight w_1 and the Lower Bound w_L

In this section, we compare the fundamental weight w_1 and its lower bound w_L in interconnected networks with different topological features. We perform all the simulations on 10^3 network realizations, respectively.

A. Interconnected clique and random network

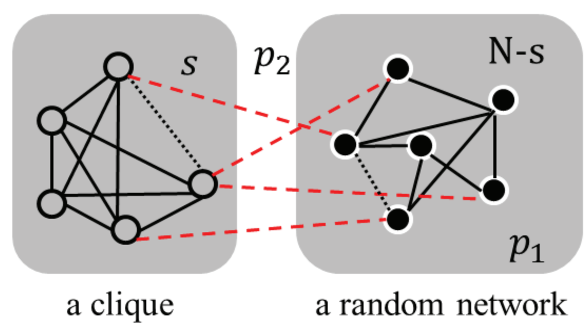

Fig. 2: *Interconnected clique and random network*

The interconnected networks here are composed of a complete graph (*i.e.* a clique) and a random network. The clique is a network in which every two nodes are connected by a link. The random network used in this work is an Erdős-Rényi (ER) graph. The ER graphs are characterized by a binomial degree distribution with $\text{Prob}\left[\widetilde{D} = k\right] = \binom{N-1}{k} {p_1}^k (1 - p_1)^{N-1-k}$, where p_1 is the probability that each node pair is connected and $\widetilde{D}$ is the degree of the nodes in the random network. The size of the clique is s and the size of the random network is $N - s$. The adjacency matrix of the interconnected complex networks can be expressed as

$$A = \begin{bmatrix} J - I & C \\ C^T & \widetilde{G} \end{bmatrix},$$

where J is the all-one matrix, I is the identity matrix. The submatrix C characterizes the interconnections between the clique and the random network. A node in the clique is connected to a node in the random network $\widetilde{G}$ with a probability p_2. If a link exists between the two nodes, the corresponding element of C is one, otherwise the element equals to zero (see Fig. 2). Note that the clique number ω_1 could be larger than s, for example, when all nodes in the clique are connected to a same node in $\widetilde{G}$.

B. Comparison of w_1 and w_L, when $p_1 = p_2$

We compare the fundamental weight w_1 and its lower bound w_L in interconnected networks of size $N = 20$ nodes. An ER random graph is connected for large N, if $p > p_c \sim \ln N/N$, where p_c is the disconnectivity threshold. In this work, the disconnectivity threshold for the random network is $\widetilde{p_c} \sim \ln(N-s)/(N-s)$. We find that the relative deviation $\frac{w_1 - w_L}{w_1}$ of the lower bound w_L increases with the increase of the interconnection probability p_2 (or p_1), when $p_1 = p_2 \leq \widetilde{p_c}$ and ω_1 is a constant value (see Fig. 3b).

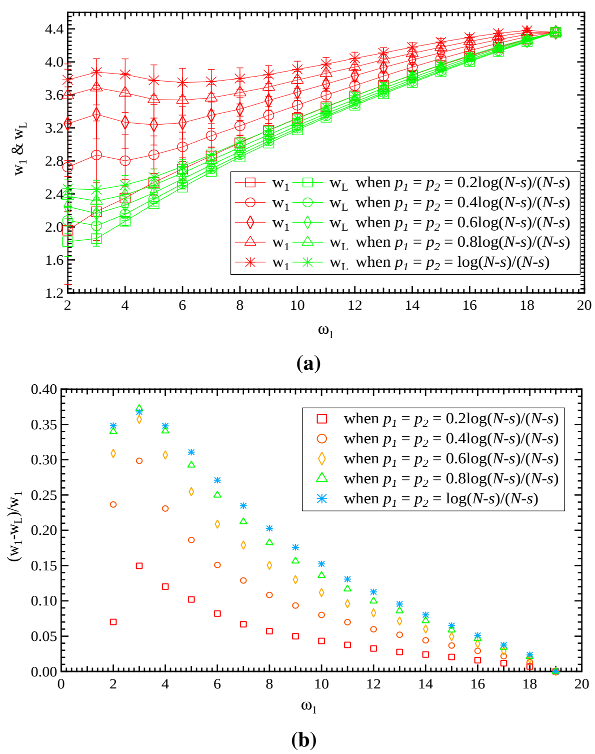

Fig. 3: *(a) Fundamental weight w_1 of the principal eigenvector and its lower bound w_L, as well as (b) the relative deviation of the lower bound as a function of the clique number ω_1 in interconnected networks with $p_1 = p_2 \leq \widetilde{p_c} = ln(N-s)/(N-s)$. The error bars for w_1 and w_L are plotted.*

However, the effect of p_1 (or p_2) is the opposite, when $p_1 = p_2 > \widetilde{p_c}$ and ω_1 is given (see Fig. 4b). When $p_1 = p_2 \to 0$ or $p_1 = p_2 \to 1$, the whole network tends to be a complete graph in which $w_1 = w_L$. This supports our observation that the relative deviation $\frac{w_1 - w_L}{w_1}$ increases with the increase of $p_1 = p_2$ when $p_1 < \widetilde{p_c}$, while decreases with the increase of $p_1 = p_2$ when $p_1 > \widetilde{p_c}$. We also find that w_1 and w_L are always closer to each other when the clique number ω_1 increases (see Figs. 3 and 4). It might be explained by the fact that the whole network tends to a complete graph where $w_1 = w_L$, when ω_1 increases and $p_1 = p_2$ is a constant value. Another interesting finding is that, when $p_1 = p_2 > \widetilde{p_c}$, w_1 is not related with the clique number ω_1 any more, and tends to be constant (see Fig. 4a). The constant value of w_1 is around $\sqrt{N}$.

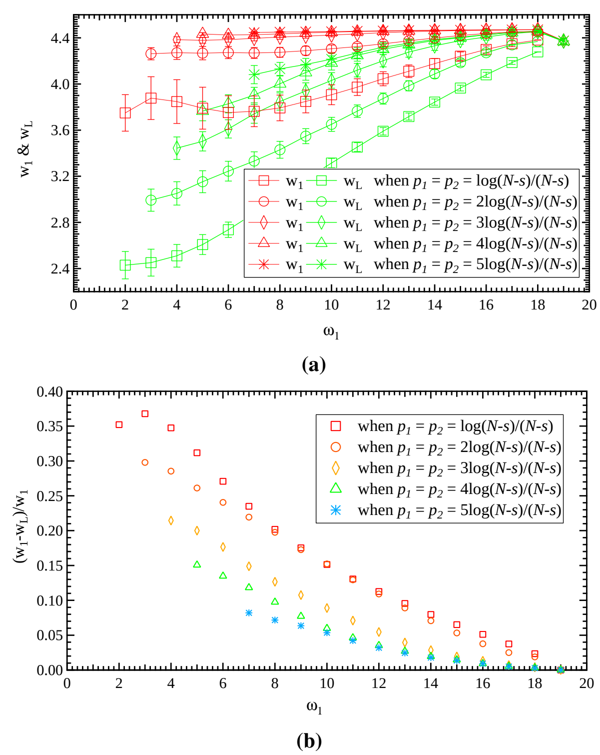

Fig. 4: *(a) Fundamental weight w_1 of the principal eigenvector and its lower bound w_L, as well as (b) the relative deviation of the lower bound as a function of the clique number ω_1 in interconnected networks with $p_1 = p_2 \geq \widetilde{p_c} = ln(N-s)/(N-s)$. The error bars for w_1 and w_L are plotted.*

C. Comparison of w_1 and w_L, when p_1 is fixed but p_2 changes

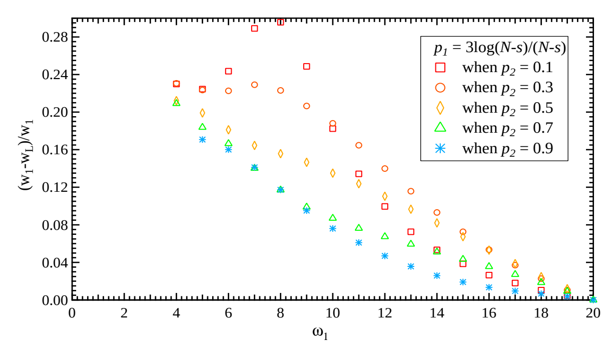

Fig. 5: *Relative deviation of the lower bound as a function of the clique number ω_1 in interconnected networks with $p_1 = 3\widetilde{p_c}$ and $p_2 = 0.1,\ 0.3, ..., 0.9$.*

Here we study the effect of the interconnection probability p_2 on the fundamental weight w_1 and its lower bound w_L. The relative deviation $\frac{w_1 - w_L}{w_1}$ decreases with the increase of the interconnection probability p_2, when $p_1 = 3\widetilde{p_c}$, in interconnected networks with the same ω_1 (see Fig. 5). The observation implies that when the largest clique is connected to more other nodes, the fundamental weight w_1 is better lower bounded by w_L.

D. Comparison of w_1 and w_L, when p_2 is fixed but p_1 changes

In this part, the fundamental weight w_1 and its lower bound w_L are studied in networks with a constant $p_2 = 0.5$. We find that the increase of the link density p_1 reduces the relative deviation $\frac{w_1 - w_L}{w_1}$, when the clique number $\omega_1 < N/2$. However, when $\omega_1 > N/2$, p_1 almost does not influence the relative deviation any more (see Fig. 6).

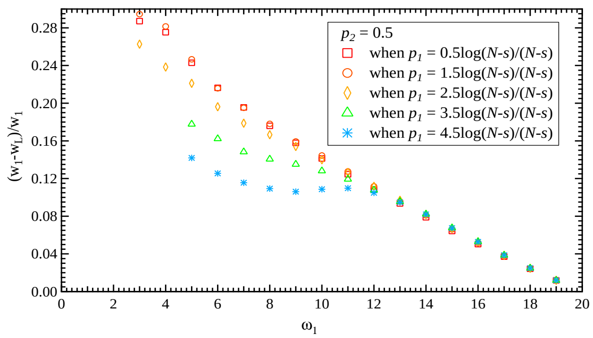

Fig. 6: *Relative deviation of the lower bound as a function of the clique number ω_1 in interconnected networks with $p_2 = 0.5$ and $p_1 = 0.5\widetilde{p_c},\ 1.5\widetilde{p_c}, ..., 4.5\widetilde{p_c}$, where $\widetilde{p_c} = ln(N-s)/(N-s)$.*

IV. Comparison in Special Cases of Interconnected Complex Networks

Here we investigate the fundamental weight w_1 and its lower bound w_L in interconnected networks with $p_1 = 0$, where $\omega_1 = s$. We denote the number of interconnections between the clique and the random network by L_c for any s value.

A. Effect of the location of the interconnections between the clique and the random network on the lower bound w_L in interconnected networks ($p_1 = 0$) with a fixed L_c

In this part we keep the number of interconnection links fixed $L_c = N - s$, but change the location of the interconnections. For example, we study the lower bound w_L in interconnected networks with $s = 2$ and $N = 20$. One node in the clique is interconnected to N_{c1} nodes in the random graph, and the other node in the clique is interconnected to the remaining $(N - s - N_{c1})$ nodes in the random graph. We find that the difference $\Delta w_1 = w_1 - w_L$ as a function of N_{c1} is almost stable when L_c is a constant (see Fig. 7). The small peak of the difference Δw_1 appears when the s nodes in the clique both have the same $(\frac{N-s}{s})$ interconnections to the nodes in the random network. When the interconnections are more evenly linked to the nodes in the clique, the maximum degree $d_{\max}$ of the interconnected network is smaller. The decrease of $d_{\max}$ could lead to the decrease of λ_1. Correspondingly, $w_L = \sqrt{\frac{\lambda_1}{1 - 1/\omega_1}}$ decreases to the minimum value, when each node in the clique is interconnected to $\frac{N-s}{s}$ nodes in the random network.

We then study the effect of $\omega_1 = s$ on the difference $\Delta w_1 = w_1 - w_L$. We still keep $L_c = N - s$ and randomly link every node in the random graph to one and only one node in the clique. We find that the relative deviation $\frac{w_1 - w_L}{w_1}$ exponentially decreases with the clique number ω_1 (see Fig. 8 inset). In this kind of networks, the maximum degree is $d_{\max} = E[D]_{\text{clique}} = (s - 1 + \frac{N-s}{s})$. Figure 8 shows that the difference $\Delta w_1 = w_1 - w_L$ is equal to zero, when $\omega_1 = s$ is sufficiently large.

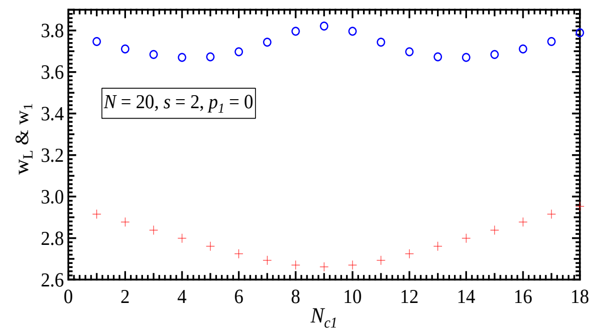

Fig. 7: *Fundamental weight w_1 of the principal eigenvector and its lower bound w_L as a function of N_{c1} in interconnected networks with the clique number $\omega_1 = s = 2$ and network size $N = 20$. The w_1 is in (blue) circle marks and the w_L is in (red) cross marks.*

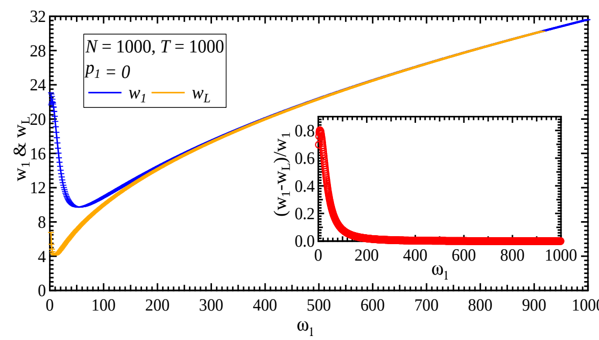

Fig. 8: *Fundamental weight w_1 of the principal eigenvector, its lower bound w_L, and the relative deviation of the lower bound as a function of the clique number ω_1 in networks ($N = 1000$). The networks contain a clique of size s and a random graph with $(N - s)$ nodes which are randomly connected to one and only one node in the clique. The simulations are performed on $T = 10^3$ realizations and the error bars for w_1 and w_L are plotted.*

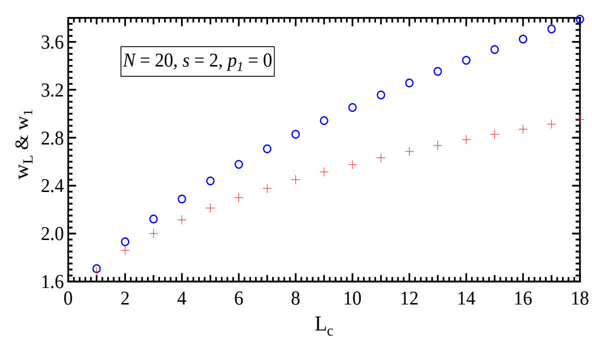

Fig. 9: *Fundamental weight w_1 of the principal eigenvector and its lower bound w_L as a function of the interconnection number L_c in interconnected networks with clique size $s = 2$ and network size $N = 20$. The w_1 is in (blue) circle marks and the w_L is in (red) cross marks.*

B. *Effect of the number L_c of interconnections between the clique and the random network ($p_1 = 0$) on the lower bound w_L*

We first investigate the influence of L_c on the lower bound w_L in interconnected networks with a fixed clique size $s = 2$ and a link density $p_1 = 0$. We find that the difference $\Delta w_1 = w_1 - w_L$ increases with the increase of the interconnection number L_c (see Fig. 9).

We next study the fundamental weight w_1 and its lower bound w_L in interconnected networks with $s = N - 1$ (see Fig. 10). We find that w_1 can be well lower bounded by w_L when $L_c \to 0$ and $L_c \to N - 1$ (see Fig. 11). This can be explained as follows: (1) when $L_c = 0$ and $L_c = N - 1$, the interconnected network can be considered as a network with separated cliques; (2) the fundamental weight w_1 of networks with separated cliques is equal to $\sqrt{\omega_1}$; and (3) the lower bound $w_L = \sqrt{\frac{\lambda_1}{1-\frac{1}{\omega_1}}}$, where $\lambda_1 = \omega_1 - 1$. Hence, $w_L = \sqrt{\omega_1} = w_1$

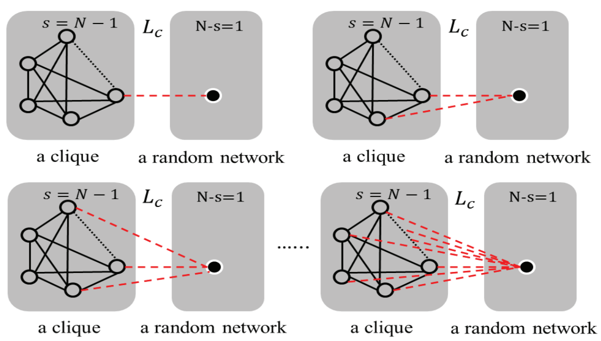

Fig. 10: *Increase of the number L_c of the interconnections between the clique ($s = N - 1$) and one node.*

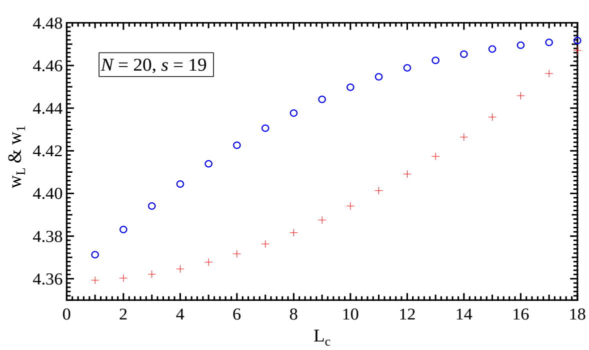

Fig. 11: *Fundamental weight w_1 of the principal eigenvector and its lower bound w_L as a function of the interconnection number L_c, in interconnected networks with clique size $s = N - 1$. The w_1 is in (blue) circle marks and the w_L is in (red) cross marks.*

In this section we have studied the fundamental weight w_1 and its lower bound $w_L = s$ in some special interconnected networks. We find that the difference $\Delta w_1 = w_1 - w_L$ is almost fixed no matter how the interconnections are placed, when the number L_c of interconnections between the clique and the random network is constant. When $L_c = N - s$, $p_1 = 0$ and the clique number ω_1 is sufficiently large, the difference $\Delta w_1 = 0$. We also observe that the difference Δw_1 first increases with the increase of the number L_c of interconnections when $L_c < \frac{s(N-s)}{2}$, and then decreases with the increase of L_c when $L_c > \frac{s(N-s)}{2}$.

V. Another Lower Bound for the Fundamental Weight w_1 of the Principal Eigenvector

The problem of finding the clique number ω_1 of a graph is an NP-hard problem. Although new algorithms have been proposed in literature [26], [27], [28], [29] to raise the calculating rate, finding the maximum clique problem is still a challenge. Here we give another fast-calculated lower bound w_s for w_1, which is not related to the clique number ω_1.

In interconnected networks introduced in Section III, the probability that the clique number ω_1 equals to the size s of the clique, is

$$\text{Prob}\left[\omega_1 = s\right] = \left(1 - {p_2}^s\right)^{N-s} \approx 1, \tag{5}$$

when $s \geq \frac{N}{2}$, and N is sufficiently large. With Eq. (5), we can reduce the lower bound w_L of the fundamental weight of x_1 to $w_s = \sqrt{\frac{\lambda_1}{1-1/s}}$, when $s \geq \frac{N}{2}$ and N is sufficiently large.

We compare the fundamental weight w_1 and its lower bound w_s in interconnected networks ($N = 20, 50, 100$). We study the influence of the network size N, the clique size s, the link density p_1 and the connecting probability p_2 on the difference between the fundamental weight w_1 and the lower bound w_s in 10^3 network realizations, respectively. We find that the deviation $\frac{w_1 - w_s}{w_1}$ decreases with the increase of s in all network realizations, but increases with the increase of p_1 and p_2 when $s \geq \frac{N}{2}$ (see Fig. 12).

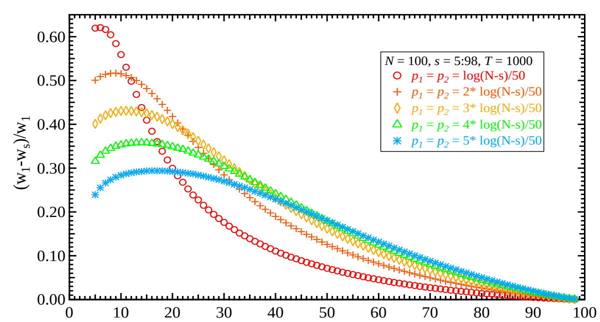

Fig. 12: *Relative deviation $\frac{w_1 - w_s}{w_1}$ of the fundamental weight as a function of the clique size s in networks ($N = 100$). The simulations are performed on 10^3 realizations.*

VI. Conclusion

We first theoretically prove that w_1 could be lower bounded by $w_L = \sqrt{\frac{\lambda_1}{1-1/\omega_1}}$ and $w_D = \sqrt{\frac{\lambda_1}{1-1/d_{max}}}$ in any network. The lower bound w_L is sharper since $w_L \geq w_D$, although its computational complexity is high. We compare the fundamental weight w_1 and the better lower bound w_L in interconnected networks which are formed by randomly interconnecting a complete network (*i.e.* a clique) with a random network. The influence of topological features, such as the link density p_1 of the random network and the interconnection probability p_2 between the nodes in the clique and the nodes in the random network, on the fundamental weight w_1 and its lower bound w_L is studied. We find that the lower bound w_L is closer to

w_1, when the clique number ω_1 increases. For networks with a same ω_1, the lower bound w_L performs better, when more nodes are connected to the largest clique. When $p_1 = p_2 \to 0$ or $p_1 = p_2 \to 1$, the lower bound $w_L \to w_1$. We next investigate the effect of the number L_c of interconnections between the clique and the random network on the quality of w_L. We find that the difference Δw_1 increases with the increase of L_c when $L_c \leq \frac{s(N-s)}{2}$, and decreases with the increase of L_c when $L_c \geq \frac{s(N-s)}{2}$. We finally propose another lower bound for w_1 when the interconnected networks we considered are large and the size of the clique $s \geq N/2$. This lower bound performs better as the clique size s increases.

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