Modeling region-based interconnection for interdependent networks

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Various real-world networks interact with and depend on each other. The design of the interconnection between interacting networks is one of the main challenges to achieve a robust interdependent network. Due to cost considerations, network providers are inclined to interconnect nodes that are geographically close. Accordingly, we propose two topologies, the random geographic graph and the relative neighborhood graph, for the design of interconnection in interdependent networks that incorporates the geographic location of nodes. Differing from the one-to-one interconnection studied in the literature, one node in one network can depend on an arbitrary number of nodes in the other network. We derive the average number of interdependent links for the two topologies, which enables their comparison. For the two topologies, we evaluate the impact of the interconnection structure on the robustness of interdependent networks against cascading failures. The two topologies are assessed on the real-world coupled Italian Internet and the electric transmission network. Finally, we propose the derivative of the largest mutually connected component with respect to the fraction of failed nodes as a robustness metric. This robustness metric quantifies the damage of the network introduced by a small fraction of initial failures well before the critical fraction of failures at which the whole network collapses.

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I. INTRODUCTION

In the real world, most networks are interdependent. For example, power networks depend on communication networks, where each node in a communication network controls one or more nodes in a power network, while each communication node needs power to function [1]. Most infrastructures are interdependent networks, such as transportation networks, communications, and energy supply networks. An interdependent network is a network consisting of different types of networks that interact with each other via interconnected links [2].

In interdependent networks, a cascade of failures leads to the first-order (discontinuous) percolation transition, whereas a second-order (continuous) phase transition characterizes the collapse of a single network [3,4]. Some types of interdependent networks also feature a structural transition [5] between distinguishable and nondistinguishable network components. The exact transition threshold for such a structural transition is determined in Ref. [6]. Most previous studies are restricted to a one-to-one interdependency between networks, where one-to-one interdependency means that one node in one network connects to one and only one node in the other network and vice versa. Boccaletti *et al.* [4] introduce models that enable nodes in one network to connect to multiple nodes in the other network, with a given degree sequence for interconnections. Moreover, the location of the nodes is not considered when designing the interconnection between interdependent networks, although connecting geographically close nodes is less costly than connecting those that are far from each other.

We propose two topologies, the random geometric graph and the relative neighborhood graph, that incorporate the location of nodes for the design of interconnection in interdependent networks. The advantages of the models are that (i) the interdependency is generalized from one-to-one to one-to-many interconnectionsand (ii) the sizes of the interdependent networks are not necessarily equal.

We derive the average number of links for the two topologies which enables the comparison between simulations performed on them. For the two topologies, we investigate the impact of the interconnection structure on the robustness of the network under node failures. The size of the largest mutually connected component (the number of functioning nodes) is employed as a robustness metric. In addition, we propose the derivative of the largest mutually connected component with respect to the fraction of failed nodes as a new robustness metric. The proposed robustness metric quantifies the damage on the whole network triggered by a small fraction of nonfunctioning nodes.

The paper is organized as follows. Section II illustrates two interconnection topologies that incorporate the location of nodes. Section III presents the cascading failures in interdependent networks. The simulation results are presented in Sec. IV and Sec. V concludes the paper.

II. REGION-BASED INTERDEPENDENCY

Consider an interdependent graph $G(N,L)$ with N nodes and L links consisting of two graphs G_1 and G_2. The adjacency matrix A of G can be written as

$$A = \begin{bmatrix} (A_1)_{n\times n} & B_{n\times m} \\ \left(B^T\right)_{m\times n} & (A_2)_{m\times m} \end{bmatrix}, \qquad (1)$$

where A_1 is the $n \times n$ adjacency matrix of the graph G_1 with n nodes, A_2 is the $m \times m$ adjacency matrix of the graph G_2 with m nodes, and B is the $n \times m$ interconnection matrix connecting G_1 and G_2. The total number of nodes in G is $N = n + m$. The interaction between networks G_1 and G_2 completely relies on the interconnection matrix B. The design of B is, therefore, crucial for the interdependent networks to function properly as a whole.

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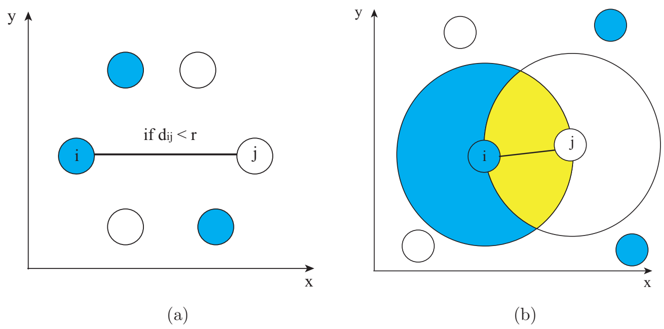

FIG. 1. Two topologies for B: (a) random geometric graph and (b) relative neighborhood graph. Since there is no third node in the intersection region (marked as yellow), nodes i and j are connected. Nodes from G_1 are represented with filled circles, whereas nodes from G_2 are represented with unfilled circles.

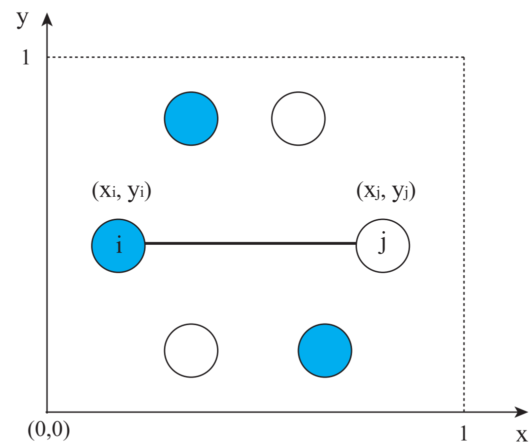

FIG. 2. Node coordinate.

In this paper, we propose two topologies for the interconnection matrix B incorporating the geographical location of nodes. Associating each node with a coordinate, we analyze the interconnection matrix B with elements b_{ij} in the following two ways: $b_{ij} = 1$ if

(1) *Random geometric graph* [7]: the Euclidean distance d_{ij} between node i and node j is smaller than a given threshold r;

(2) *Relative neighborhood graph* [8]: there is no third node in the intersection region of two circles with centers at nodes i and j with the same radius equal to their Euclidean distance d_{ij}.

Figure 1 shows the two topologies of the interconnection matrix B.

A. Random geometric graph

A random geometric graph, denoted as $G_{p_{ij}}(N)$, consists of N nodes and two nodes i and j are connected by a link with probability p_{ij}. Consider N independent and identically distributed nodes in a two-dimensional square with size Z. Any square with size Z can be normalized [9] to a unit square ($Z = 1$) without changing the probability p_{ij}. For simplicity, we consider a unit square with size $Z = 1$. Let (x_i, y_i) and (x_j, y_j) be the coordinates for nodes i and j as illustrated in Fig. 2. Let $r \geqslant 0$ be a non-negative and real number which is referred to as the radius of a node. The probability $p_{ij}(r) = \Pr[d_{ij} \leqslant r]$ is the probability that the Euclidean distance $d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ between two uniformly distributed nodes i and j is less than or equal to the radius r. The maximum Euclidean distance between two nodes in a two-dimensional square with size $Z = 1$ is $\sqrt{2}$. When $r \geqslant \sqrt{2}$, the probability for nodes i and j being connected is $p_{ij} = 1$ and, thus, the graph $G_{p_{ij}}(N)$ is a complete graph K_N.

In Sec. II A 1, we prove a theorem for p_{ij} in a general random geometric graph in a two-dimensional square with size $Z = 1$.

1. *Probability p_{ij} of having a link between nodes i and j*

Theorem 1. The probability $p_{ij}(r)$ that there is a link l_{ij} between nodes i and j in a random geometric graph in a two-dimensional unit square is

$$p_{ij}(r) = \begin{cases} \pi r^2 - \frac{8}{3}r^3 + \frac{1}{2}r^4 & 0 \leqslant r \leqslant 1 \\ \frac{1}{6}\{-3r^4 + (16r^2+8)\sqrt{r^2-1} + 12r^2[\arctan(\frac{2-r^2}{2\sqrt{r^2-1}}) - 1] + 2\} & 1 \leqslant r \leqslant \sqrt{2} \end{cases}.$$

Proof. The probability $p_{ij}(r)$ that there is a link l_{ij} between nodes i and j in a square with size $Z = 1$ is

$$p_{ij}(r) = \Pr[(x_i - x_j)^2 + (y_i - y_j)^2 \leqslant r^2].$$

Let $Z_1 = |X_1 - X_2|$ and $Z_2 = |Y_1 - Y_2|$ be random variables. The probability distribution function for Z_1 is, when $0 \leqslant z_1 \leqslant 1$,

$$F(z_1) = \Pr[-z_1 \leqslant X_1 - X_2 \leqslant z_1].$$

Since X_1 and X_2 are independent uniform random variables, we obtain

$$\Pr[X_1 - X_2 \leqslant z_1] = \int_0^{1-z_1}\int_0^{x_2+z_1} dx_1 dx_2 + \int_{1-z_1}^1 \int_0^1 dx_1 dx_2 = \frac{1}{2}(1 - z_1^2) + z_1.$$

Analogously,

$$\Pr[X_1 - X_2 \leqslant -z_1] = \tfrac{1}{2}(z_1 - 1)^2.$$

With $F(z_1) = \Pr[X_1 - X_2 \leqslant z_1] - \Pr[X_1 - X_2 \leqslant -z_1]$, we arrive at

$$F(z_1) = -z_1^2 + 2z_1.$$

The probability density function $f(z_1) = F'(z_1)$ follows, when $0 \leqslant z_1 \leqslant 1$,

$$f(z_1) = 2(1 - z_1).$$

Since Z_1 and Z_2 are independent and identically distributed, we have

$$\Pr\left[(x_i - x_j)^2 + (y_i - y_j)^2 \leqslant r^2\right] = \iint_{{z_1}^2+{z_2}^2 \leqslant r^2} f(z_1)f(z_2)dz_1dz_2.$$

For $0 \leqslant r \leqslant 1$, we have, after transformation to polar coordinates,

$$\Pr[(x_i - x_j)^2 + (y_i - y_j)^2 \leqslant r^2] = \int_0^r \int_0^{\sqrt{r^2 - z_2^2}} f(z_1)f(z_2)dz_1dz_2 = \pi r^2 - \frac{8}{3}r^3 + \frac{1}{2}r^4. \tag{2}$$

Similarly, we find, for $1 < r \leqslant \sqrt{2}$,

$$\begin{aligned}\Pr[(x_i - x_j)^2 + (y_i - y_j)^2 \leqslant r^2] &= \int_0^1 \int_0^{\sqrt{r^2-1}} f(z_1)f(z_2)dz_1dz_2 + \int_{\sqrt{r^2-1}}^1 \int_0^{\sqrt{r^2 - z_1^2}} f(z_1)f(z_2)dz_1dz_2 \\ &= \frac{1}{3} + 2r^2\left[\arctan\left(\frac{2-r^2}{2\sqrt{r^2-1}}\right) - 1\right] + \frac{8r^2+4}{3}\sqrt{r^2-1} - \frac{1}{2}r^4.\end{aligned} \tag{3}$$

Combining (2) and (3) establishes Theorem 1. ■

Figure 3 shows the probability p_{ij} as a function of the radius r in a random geometric graph $G_{p_{ij}}(N)$ with $N = 10^4$ nodes. The simulation shows an excellent agreement with Theorem 1.

From Theorem 1, the average number of links for a random geometric graph with N nodes is $E[L] = \binom{N}{2} p_{ij}(r)$.

B. Relative neighborhood graph

A relative neighborhood graph, denoted as $RNG(N)$, consists of N nodes and two nodes i and j are connected if $d_{ij} \leqslant \max(d_{ik}, d_{jk})$ for all the other nodes $k = 1,2,\ldots,N$, $k \neq i,j$. Figure 4 shows a set of N nodes in a two-dimensional square with size $Z = 1$ and its relative neighborhood graph. In Sec. II B 1, we prove a theorem for the lower bound of the probability p_{ij} of nodes i and j being connected in a general relative neighborhood graph.

1. *Probability p_{ij} of having a link between nodes i and j*

Theorem 2. The probability p_{ij} that for a relative neighborhood graph there is a link l_{ij} between nodes i and j in a two-dimensional square with size $Z = 1$ is lower bounded by

$$p_{ij} \geqslant \frac{\pi c N + 1}{c^2 N(N-1)} - \frac{2\sqrt{\pi}\,\Gamma(N-1)}{c^{\frac{3}{2}}\Gamma\left(N + \frac{1}{2}\right)}, \tag{4}$$

where $c = (\frac{2\pi}{3} - \frac{\sqrt{3}}{2})$ and $\Gamma(x)$ is the gamma function.

Proof. Given a pair of nodes i and j uniformly distributed in the square with size $Z = 1$, let A be the random variable for the area of the intersection region [marked as yellow in Fig. 1(b)] of two circles centered at nodes i and j and with d_{ij} as the radius. For a two-dimensional square with size $Z = 1$, the area of the square is 1. The probability p_{ij} that nodes i and j being connected equals the probability that all the other $N - 2$ nodes are not in the intersection region A:

$$p_{ij} = (1 - A)^{N-2}.$$

Using the law of total probability [10], we have

$$p_{ij} = \int_0^1 (1 - x)^{N-2} f_A(x)dx, \tag{5}$$

where $f_A(x)$ is the probability density function of A. The probability distribution function for the variable A is

$$F_A(x) = \Pr[A \leqslant x].$$

Let D be the random variable of the distance between two nodes. The area [11] of the intersection of two circles

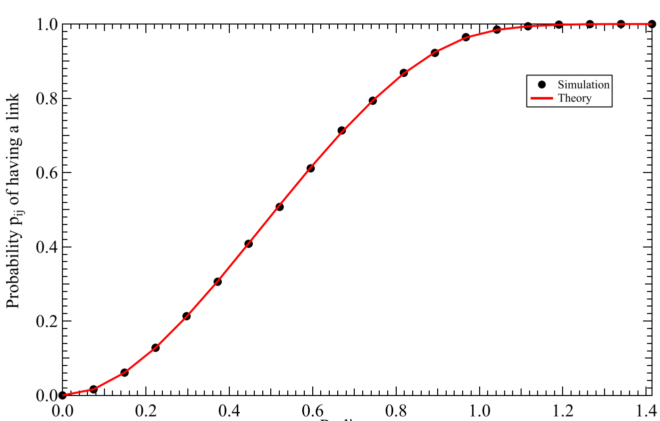

FIG. 3. The probability $p_{ij}(r)$ that nodes i and j are connected as a function of the radius r in a random geometric graph with $N = 10^4$ nodes.

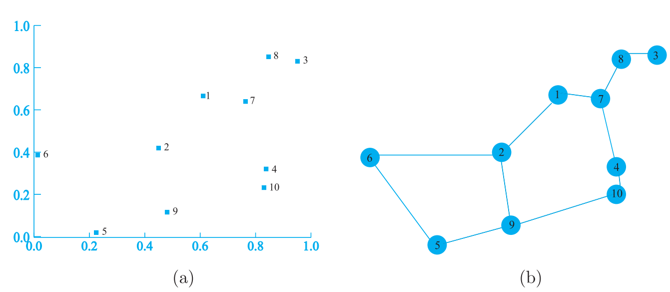

FIG. 4. An example of (a) a set of N nodes and its (b) relative neighborhood graph.

can be computed by D^2c, where $c = (\frac{2\pi}{3} - \frac{\sqrt{3}}{2})$. When the intersection is completely in the two-dimensional unit square, it holds that $A = D^2c$. When the intersection is partially in the unit square, we have, for $\varepsilon > 0$, that $A + \varepsilon = D^2c$ and, hence,

$$F_A(x) = \Pr[D^2c - \varepsilon \leqslant x] \geqslant \Pr[D^2c \leqslant x].$$

Applying $D^2 = (x_i - x_j)^2 + (y_i - y_j)^2$ and $r^2 = \frac{x}{c} < 1$ in (2) yields

$$\Pr[D^2c \leqslant x] = \frac{\pi x}{c} - \frac{8}{3}\left(\frac{x}{c}\right)^{\frac{3}{2}} + \frac{1}{2}\left(\frac{x}{c}\right)^2.$$

The probability distribution function is lower bounded by

$$F_A(x) \geqslant \frac{\pi x}{c} - \frac{8}{3}\left(\frac{x}{c}\right)^{\frac{3}{2}} + \frac{1}{2}\left(\frac{x}{c}\right)^2$$

from which

$$f_A(x) \geqslant \frac{\pi}{c} - 4\left(\frac{x}{c^3}\right)^{\frac{1}{2}} + \frac{x}{c^2}.$$

Thus, we have for (5)

$$p_{ij} \geqslant \int_0^1 (1-x)^{N-2}\left[\frac{\pi}{c} - 4\left(\frac{x}{c^3}\right)^{\frac{1}{2}} + \frac{x}{c^2}\right]dx. \quad (6)$$

Using the Beta function $B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ in (6), we establish Theorem 2. ■

It has been shown [8] that the relative neighborhood graph is a superset of the minimum spanning tree. The number L of links in the relative neighborhood graph with N nodes is bounded [8] by

$$N - 1 \leqslant L \leqslant 3N - 6. \quad (7)$$

Hence, the link density $p = \frac{L}{\binom{N}{2}}$ for a relative neighborhood graph is bounded by $\frac{2}{N} \leqslant p \leqslant \frac{6(N-2)}{N(N-1)}$, which shows that the relative neighborhood graph is a sparse graph: the larger the size N of the graph, the sparser the graph is. From Theorem 2, we deduce the lower bound for the average number $E[L]$ of links

$$E[L] \geqslant \binom{N}{2} p_{ij}. \quad (8)$$

A different lower bound for $E[L]$ is presented in Ref. [12],

$$E[L] \geqslant 0.689N. \quad (9)$$

Figure 5 shows the average number of links $E[L]$ for RNG(N) with N ranging from 50 to 200. Figure 5 shows that our bound (8) is close to the simulations and outperforms bound (9).

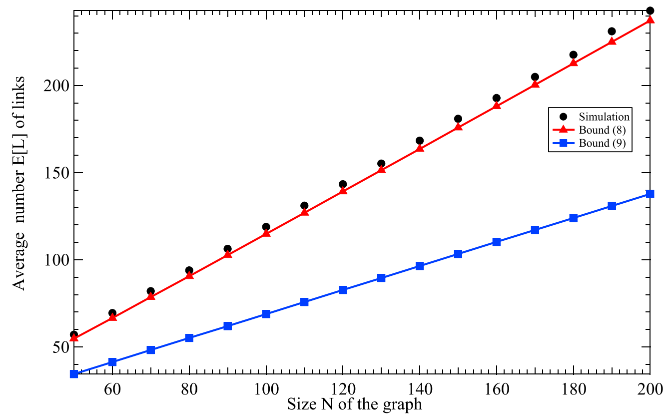

FIG. 5. Number of links for $RNG(N)$ with N ranging from 50 to 200.

III. CASCADING FAILURES IN INTERDEPENDENT NETWORKS

When nodes in one network fail, the interconnection structure between two networks causes dependent nodes in the other network also to fail. This may happen recursively and may invoke a cascading failure until no more nodes fail. In this section, we investigate the impact of interconnection topologies on the robustness of interdependent networks against cascading failures. The robustness is quantified by (i) the Largest Mutually Connected Component (LMCC) and (ii) a derivative of the largest mutually connected component with respect to the fraction of removed nodes.

A. Largest mutually connected component

Differing from the models [3,13,14] where a node from one network depends on one and only one node from the other network (one-to-one interconnection), we generalize the interconnection pattern to one-to-many: A node might depend on zero or one or more than one node depending on the distance to other nodes.

In our model, we assume a node n_1 in network G_1 to be functional if (i) its interdependent nodes in network G_2 are functioning and (ii) the node belongs to the giant component of the functional nodes in network G_1. Since a node n_1 in G_1 may have more than one support node in G_2, we assume two scenarios for n_1 being supported by nodes in G_2: (i) *at least one* of the supported nodes in G_2 is functioning and (ii) *all* of its supported nodes in G_2 are functioning. The same assumptions are applied to the nodes in network G_2.

A random removal of a fraction $1-q$ of nodes in network G_1, on one hand, isolates nodes in network G_1 and, on the other hand, causes nodes in network G_2 to fail because of the removed interconnected nodes in G_1. The failed nodes in network G_2 isolate nodes from the giant component in networks G_2. The isolated nodes in G_2 further introduce failures in G_1 and so on. The cascading failures continue until no more nodes fail. The remaining set of functional nodes is referred to as the LMCC. We assume, without loss of generality, that the fraction $1-q$ of nodes is removed from graph G_1.

1. Algorithm description

The metacode for computing the largest mutually connected component is given in Algorithm 1. The main algorithm starts at line 3 where n is the number of realizations of G. Lines 4 to 16 generate an interdependent graph G consisting of either two Erdős-Rényi (ER) graphs or two Barabási-Albert (BA) graphs. The interconnection topology is either the random geometric graph (RGG in line 12) or the relative neighborhood graph (RNG in line 14). From line 17 to line 22, we compute the largest mutually connected component

Algorithm 1. AverageLMCC

1: **Input**: Sizes N_1 and N_2 for graphs G_1 and G_2, respectively; The parameter graph specifies G_1 and G_2 to be ER graphs with link density p or BA graphs with m; The parameter interconnection specifies B to be RGG with radius r or RNG.
2: **Output**: Average of the LMCC over n graph instances.
3: **for** i=1 **to** n **do**
4:   **if** graph = ER **then**
5:     $G_1 \leftarrow \mathrm{ER}(N_1,p)$ {generate an ER graph where nodes are connected with probability p }
6:     $G_2 \leftarrow \mathrm{ER}(N_2,p)$
7:   **else if** graph = BA **then**
8:     $G_1 \leftarrow \mathrm{BA}(N_1,m)$ {generate a BA graph where a new node with m links preferentially connects to high degree nodes}
9:     $G_2 \leftarrow \mathrm{BA}(N_2,m)$
10:   **end if**
11:   **if** interconnection = RGG **then**
12:     $B \leftarrow \mathrm{RGG}(N_1,N_2,r)$ {$N_1 \times N_2$ interconnection matrix where $B_{ij} = 1$ if $d_{ij} < r$ }
13:   **else if** interconnection = RNG **then**
14:     $B \leftarrow \mathrm{RNG}(N_1,N_2)$ {$N_1 \times N_2$ interconnection matrix where $B_{ij} = 1$ if $d_{ij} \leqslant \max(d_{ik},d_{jk})$ for all $k = 1,2,\ldots,N, k \neq i,j$}
15:   **end if**
16:   $G \leftarrow \begin{bmatrix} G_1 & B \\ B^T & G_2 \end{bmatrix}$
17:   $\mathcal{N}_1 \leftarrow$ node labels of G_1 in G
18:   $\mathcal{N}_2 \leftarrow$ node labels of G_2 in G
19:   **for** $1-q=0$ **to** $\frac{N_1}{N}$ **step** 0.01 **do**
20:     $endGraph \leftarrow \mathrm{CASCADING}(G,1-q,\mathcal{N}_1,\mathcal{N}_2)$
21:     $\mathrm{T}_{1-q} \leftarrow |\mathrm{COMPONENT}(endGraph,\mathcal{N}_1,\mathcal{N}_2)|$
22:   **end for**
23:   $\mathrm{LMCC}[i] \leftarrow T$
24: **end for**
25: **return** mean(LMCC)

after cascading failures triggered by $1-q$ removals. Lines 23 and 25 average the largest mutually connected component over n instances of G. The metacode for function CASCADING (line 20) and COMPONENT (line 21) is given in Appendix A.

We elaborate on two special values of $1-q$, i.e., 0 and $\frac{N_1}{N}$. For $1-q=0$, we assume LMCC = 1. We encounter a special scenario that there exists nodes without supporting nodes before any removals, as shown in Fig. 6, due to their location being far from nodes in the other network. We assume such nodes are alive until they are isolated from their own network. When $1-q=\frac{N_1}{N}$, the nodes in graph G_1 are completely removed. Nodes in G_2 have no supporting nodes from G_1 and thus also fail. Hence, there is no largest mutually connected component and LMCC = 0.

Figure 7 exemplifies Algorithm 1 when G_1 and G_2 are complete graphs and the interconnection matrix is $B = J$, where J is the all one matrix representing all-to-all interconnections. We assume that a node is alive if at least one of its supporting nodes is alive. Figure 7 shows that when $1-q=0$, the interdependent network is fully connected and LMCC = 1. With the increase of $1-q$ removals, LMCC decreases linearly with $1-q$. The slope of the line is -1. When $1-q=\frac{N_1}{N}$ (0.5 in Fig. 7), the nodes in graph G_1 are completely removed and LMCC = 0.

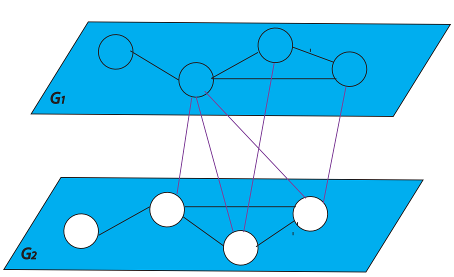

FIG. 6. An interdependent network with nodes having no interconnected nodes.

B. Derivative for the largest mutually connected component

In the real world, a network that completely collapses is a disaster for network providers. To avoid the disaster, under-

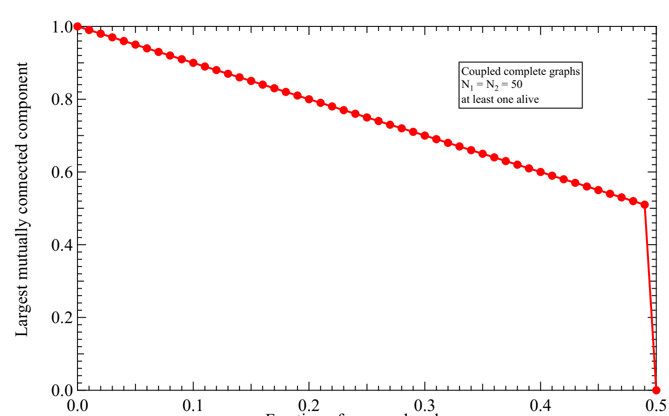

FIG. 7. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are complete graphs and the interconnection matrix is $B = J$.

standing the impact of the failure of a relatively small fraction, e.g., 10%, of nodes is significant for network providers. We theoretically approach the robustness of interdependent networks under a small fraction of failures by investigating the derivative of the largest mutually connected component close to $1-q=0$. We suggest that this derivative can be used as a robustness measure of a network indicating the extent of damage on networks when a small fraction of nodes initially fails. The smaller the absolute derivative is, the higher robustness the network exhibits.

Starting from the derivative in a single network in Sec. III B 1, we move step by step towards the derivative in interdependent networks with one-to-many interconnection in Sec. III B 3.

1. Derivative of the largest connected component for a single network

Given the probability generating function $\varphi_D(z)$ of the degree D of an arbitrary node, the probability generating function $\varphi_{(D_{l^+}-1)}$ of the degree of an end node l^+ reached by following an arbitrarily chosen link l is $\frac{\varphi'_D(z)}{\varphi'_D(1)}$, see Ref. [10]. Let $\varphi_{C_{l^+}}(z)$ be the generating function of the size C_{l^+} of components that are reached by following a random link l towards one of its end nodes l^+. If we choose a random node n in G and let $n=l^-$, then we reach a component with generation function $\varphi_{C_n}(z)$ by following the link l towards the other end node l^+. If a node in the graph is occupied uniformly at random with probability q, then the probability generating functions $\varphi_{C_{l^+}}(z)$ and $\varphi_{C_n}(z)$ follow [10]

$$\varphi_{C_{l^+}}(z) = 1-q+qz\varphi_{(D_{l^+}-1)}[\varphi_{C_{l^+}}(z)]$$

$$\varphi_{C_n}(z) = 1-q+qz\varphi_D[\varphi_{C_{l^+}}(z)].$$

Let S be the fraction of nodes in the largest connected component. Since $\varphi_{C_n}(z)$ generates the probability distribution of C_n excluding the giant component and with $\varphi_{C_n}(1)=1$, we have that [10]

$$S = 1-\varphi_{C_n}(1) = q - q\varphi_D[\varphi_{C_{l^+}}(1)],$$

where

$$\varphi_{C_{l^+}}(1) = 1-q+q\varphi_{(D_{l^+}-1)}[\varphi_{C_{l^+}}(1)]. \tag{10}$$

The derivative of the largest connected component S with respect to q is

$$\frac{dS}{dq} = 1-\varphi_D(u) - q\varphi'_D(u)u',$$

where $u=\varphi_{C_{l^+}}(1)$. The derivative of (10) follows

$$u' = \varphi_{(D_{l^+}-1)}(u) + q\varphi'_{(D_{l^+}-1)}(u)u' - 1.$$

Combining $\frac{u+q-1}{q} = \varphi_{(D_{l^+}-1)}(u) = \frac{\varphi'_D(z)}{\varphi'_D(1)}$ and $\varphi'_D(1)=E[D]$, we arrive at

$$\frac{dS}{dq} = \frac{S}{q} - \frac{E[D](u-1)(u-1+q)}{q[1-q\varphi'_{(D_{l^+}-1)}(u)]}. \tag{11}$$

When graph G is a large ER random graph, there holds to a good approximation [10, p. 39] that $\varphi_D(z) = \varphi_{(D_{l^+}-1)}(z) = e^{E[D](z-1)}$. In that case, the derivative $\frac{dS}{dq}$ in (11) can be simplified, with $u=1-S$, to

$$\frac{dS}{dq} = \frac{S}{q(1-E[D](q-S))}.$$

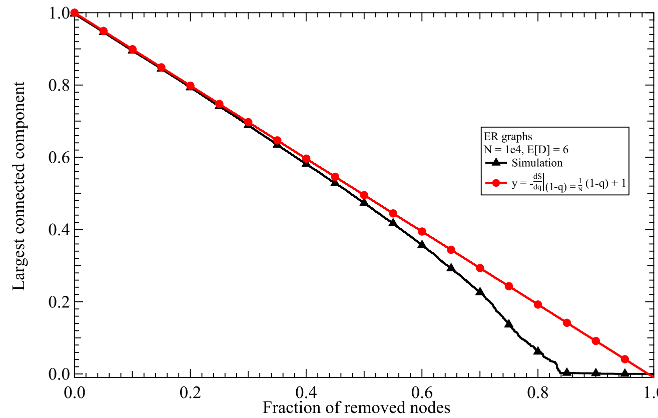

FIG. 8. Largest connected component as a function of the fraction of removed nodes in Erdős-Rényi graphs $G_p(N)$.

Figure 8 shows the straight line $y = -\frac{dS}{dq}|_{1-q=\frac{1}{N}}(1-q)+1$ and simulations of the largest mutually connected component. The straight line with slope $-\frac{dS}{dq}|_{1-q=\frac{1}{N}}$ shows a good estimation for the largest mutually connected component when a small fraction $1-q$ of nodes is removed.

2. Derivative for interdependent networks with one-to-one interconnection

Let $u_A = \varphi_{C_{l^+}}(1)$ for graph G_1 and $u_B = \varphi_{C_{l^+}}(1)$ for graph G_2. For interdependent networks with one-to-one interconnection, we have

$$u_A = \varphi_{(D_{l^+}-1)}[1-q(1-u_B)(1-u_A)]. \tag{12}$$

Analogously,

$$u_B = \varphi_{(D_{l^+}-1)}[1-q(1-u_A)](1-u_B).$$

A randomly chosen node in G_1 belongs to the largest mutually connected component if (i) the node is occupied with probability q, (ii) the node with probability $1-\varphi_{C_{G_1}}(1)$ belongs to the giant component in G_1, and (iii) the corresponding dependent node with probability $1-\varphi_{C_{G_2}}(1)$ belongs to the giant component in G_2. When graphs G_1 and G_2 are two large ER random graphs $G_p(N)$ with approximate Poisson degree distribution, we have $\varphi_D(z) = \varphi_{(D_{l^+}-1)}(z) = e^{E[D](z-1)}$. Thus, $\varphi_{C_{G_1}}(1) = u_A$ and $\varphi_{C_{G_2}}(1) = u_B$. The fraction S of nodes in the largest mutually connected component follows

$$S = q(1-u_A)(1-u_B), \tag{13}$$

where

$$u_A = e^{-qE[D](u_A-1)(u_B-1)},$$
$$u_B = e^{-qE[D](u_A-1)(u_B-1)}. \tag{14}$$

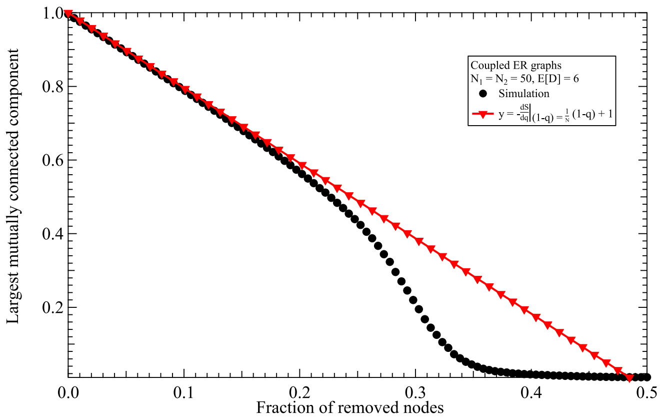

FIG. 9. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Erdős-Rényi random graphs $G_p(N)$ with $N = 50$ and the average degree $E[D] = 6$. The interdependency is one to one. The results are averaged over 10^4 realizations of interdependent graphs.

The derivative of the largest mutually connected component with respect to q in (13) is

$$\frac{dS}{dq} = (1-u_A)(1-u_B) - q\left[(1-u_B)\frac{du_A}{dq} + (1-u_A)\frac{du_B}{dq}\right].$$

The derivative for u_A in (12) follows as

$$\frac{du_A}{dq} = \frac{-\varphi'_{(D_{l^+}-1)}(u_A)(1-u_A)^2}{1-2q(1-u_A)\varphi'_{(D_{l^+}-1)}(u_A)}.$$

For ER random graphs, we have that $\varphi'_{(D_{l^+}-1)}(u_A) = E[D]u_A$. Thus,

$$\frac{du_A}{dq} = \frac{-E[D]u_A(1-u_A)^2}{1-2qu_A(1-u_A)E[D]}.$$

With $(1-u_A)(1-u_B) = \frac{S}{q}$ and $u_A = u_B$ from (14), we arrive at

$$\frac{dS}{dq} = \frac{S}{q(1-2E[D](\sqrt{Sq}-S))}. \tag{15}$$

Figure 9 shows the straight line $y = -\frac{dS}{dq}|_{1-q=\frac{1}{N}}(1-q)+1$ with slope computed from (15) and simulations of the largest mutually connected component for coupled ER random graphs $G_p(N)$. Again, the straight line with slope $-\frac{dS}{dq}|_{1-q=\frac{1}{N}}$ shows a good estimation for the largest mutually connected component when a small fraction $1-q$ of nodes is removed.

3. *Fraction of largest mutually connected component with one-to-many interconnections*

Assume that a node is alive if at least one of its interdependent nodes is alive. Theorem 3 presents the fraction S_1 and S_2 of the largest mutually connected component for network G_1 and G_2, respectively.

Theorem 3. Consider an interdependent network consisting of two graphs G_1 and G_2. The interconnection topology between graphs G_1 and G_2 is the random geometric graph. The fraction S_i $(i = 1,2)$ of the largest mutually connected component as a function of $1-q$ removals is approximated by

$$S_1 = q\left[1-\varphi_{C_{G_1}}(1)\right]\{1-(1-p_{ij})^{[1-\varphi_{C_{G_2}}(1)]N}\}, \tag{16}$$

$$S_2 = \left[1-\varphi_{C_{G_2}}(1)\right][1-(1-p_{ij})^{q[1-\varphi_{C_{G_1}}(1)]N}], \tag{17}$$

with

$$\varphi_{C_{G_1}}(1) = \varphi_{D_{G_1}}\{1-q[1-(1-p_{ij})^{(1-u_B)N}](1-u_A)\}$$
$$\varphi_{C_{G_2}}(1) = \varphi_{D_{G_2}}\{1-[1-(1-p_{ij})^{q(1-u_A)N}](1-u_B)\}$$

and

$$u_A = \varphi_{(D_{l^+}-1)}\{1-q[1-(1-p_{ij})^{(1-u_B)N}](1-u_A)\}$$
$$u_B = \varphi_{(D_{l^+}-1)}\{1-[1-(1-p_{ij})^{q(1-u_A)N}](1-u_B)\},$$

where p_{ij} is the probability that there is a link l_{ij} between node i in graph G_1 and node j in graph G_2. $1-\varphi_{C_{G_1}}(1)$ is the fraction of nodes belonging to the giant component in graph G_1 and $1-\varphi_{C_{G_2}}(1)$ in graph G_2.

Proof. For network G_1, a node i is occupied with probability q. The node i is supported with at least one node with probability $1-(1-p_{ij})^{(1-u_B)N}$, where $(1-p_{ij})^{(1-u_B)N}$ is the probability that node i does not connect to any nodes in the giant component in graph G_2. Therefore, (12) is modified to

$$u_A = \varphi_{(D_{l^+}-1)}\{1-q[1-(1-p_{ij})^{(1-u_B)N}](1-u_A)\}.$$

Analogously, for network G_2,

$$u_B = \varphi_{(D_{l^+}-1)}\{1-[1-(1-p_{ij})^{q(1-u_A)N}](1-u_B)\}.$$

Since we do not remove nodes from graph G_2 at the beginning of the removal, nodes in graph G_2 are occupied with probability 1. After cascading failures, a node in G_1 is in the largest mutually connected component if (i) the node is occupied with probability q, (ii) the node with probability $1-\varphi_{C_{G_1}}(1)$ belongs to the giant component in G_1, and (iii) at least one of the corresponding dependent node with probability $\{1-(1-p_{ij})^{[1-\varphi_{C_{G_2}}(1)]N}\}$ belongs to the giant component in G_2. A node in G_2 is in the largest mutually connected component if (i) the node with probability $1-\varphi_{C_{G_2}}(1)$ belongs to the giant component in G_2 and (ii) at least one of the corresponding dependent nodes with probability $[1-(1-p_{ij})^{q[1-\varphi_{C_{G_1}}(1)]N}]$ belongs to the giant component in G_1. ■

When graphs G_1 and G_2 are two large ER random graphs with $\varphi_D(z) = \varphi_{(D_{l^+}-1)}(z) = e^{E[D](z-1)}$, (16) and (17) can be simplified to

$$S_1 = q(1-u_A)[1-(1-p_{ij})^{(1-u_B)N}], \tag{18}$$
$$S_2 = (1-u_B)[1-(1-p_{ij})^{q(1-u_A)N}], \tag{19}$$

with

$$u_A = e^{E[D_1]q[1-(1-p_{ij})^{(1-u_B)N}](u_A-1)},$$
$$u_B = e^{E[D_2][1-(1-p_{ij})^{q(1-u_A)N}](u_B-1)}. \tag{20}$$

Figures 10(a) and 10(b) show the simulation results and S_1 and S_2 in (18) and (19) in coupled ER graphs with interconnection of random geometric graph with radius $r = 0.2$.

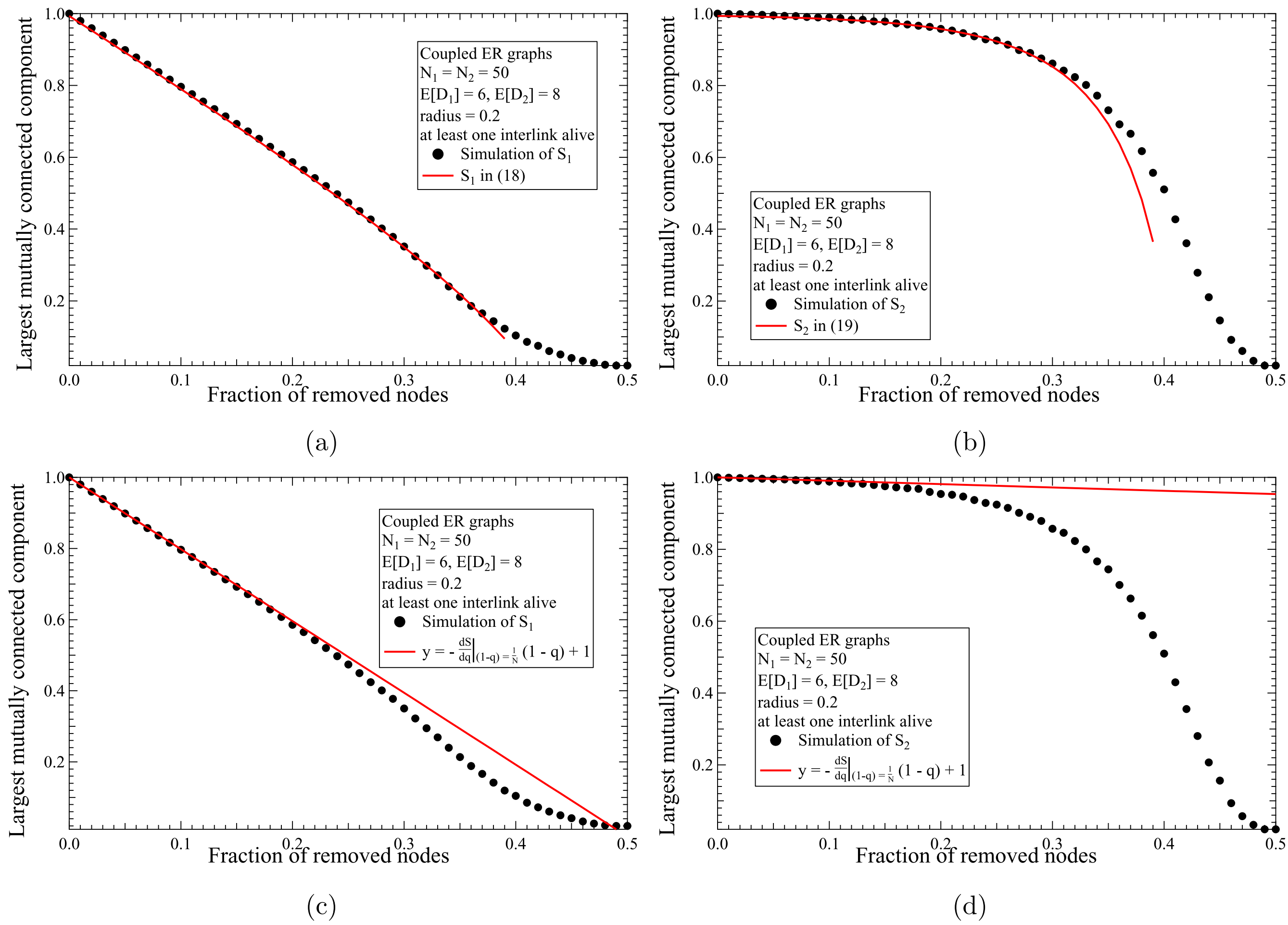

FIG. 10. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Erdős-Rényi graphs $G_p(N)$ with $N = 50$ and the average degrees $E[D_1] = 6$ and $E[D_2] = 8$. The interconnection topology is the random geometric graph with $r = 0.2$. The results are averaged over 10^3 realizations of interdependent graphs.

Since u_A and u_B are functions of q, computing the derivatives of u_A and u_B with respect to q in (20) is complicated. The derivatives of S_1 and S_2 with respect to q in (18) and (19) are even more complex. Therefore, we numerically compute the derivative $\frac{dS_i}{dq}$ $(i = 1,2)$ based on (18) and (19). Figures 10(c) and 10(d) show the simulation results and a straight line $y = -\frac{dS_i}{dq}|_{1-q=\frac{1}{N}}(1-q) + 1 (i = 1,2)$. In Figs. 10(c) and 10(d), the straight line with slope $-\frac{dS_i}{dq}(i = 1,2)$ obtained from Theorem 3 shows a good approximation for the simulations for a small fraction of removals.

For the assumption that a node is alive if all its dependent nodes are alive, the results are given in Appendix B.

IV. SIMULATION RESULTS

In this section, we investigate the impact of two interconnection topologies, the random geometric graph and the relative neighborhood graph, on the robustness of interdependent networks against cascading failures. The robustness is quantified by the LMCC when a fraction $1 - q$ of nodes are removed.

We simulate a twofold interdependent network consisting of two ER graphs $G_p(N)$ or two BA graphs. We consider two scenarios for a node being supported by the coupled network: (i) at least one dependent node alive and (ii) all the dependent nodes alive. Each node has randomly assigned coordinates $0 \leqslant x_i \leqslant 1$ and $0 \leqslant y_i \leqslant 1$.

A. Random geometric graph as interconnection

The interconnection topology between two graphs is the random geometric graph with radius r. Figure 11 shows the largest mutually connected component as a function of the fraction $1 - q$ of the removed nodes from G_1. The interdependent network consists of two Erdős-Rényi graphs $G_p(N)$ with $N = 50$ and the average degree $E[D] = 6$. We assume a node is supported by its interconnected nodes when at least one of the interconnected nodes is alive.

For a given radius r, the LMCC in Fig. 11 first decreases almost linearly with the increase of the fraction of removed nodes. Then, the LMCC experiences a first-order phase transition which differs from second-order phase transition in a single network also observed in Ref. [3] with one-to-one interconnection. Moreover, the largest mutually connected

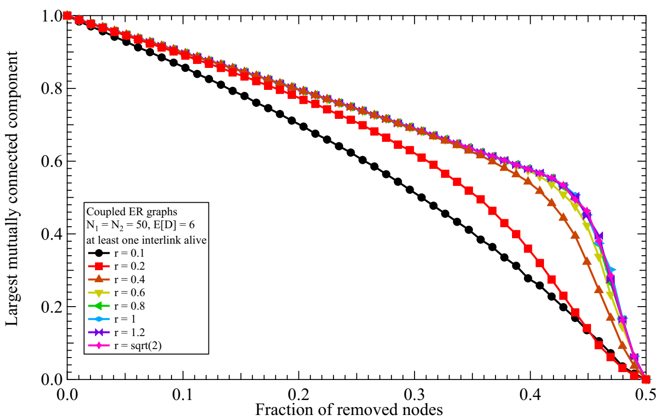

FIG. 11. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Erdős-Rényi graphs $G_p(N)$ with $N = 50$ and the average degree $E[D] = 6$. The radius r in the random geometric graph is ranging from 0.1 to $\sqrt{2}$. The simulations are averaged over the results from 1000 interdependent graphs.

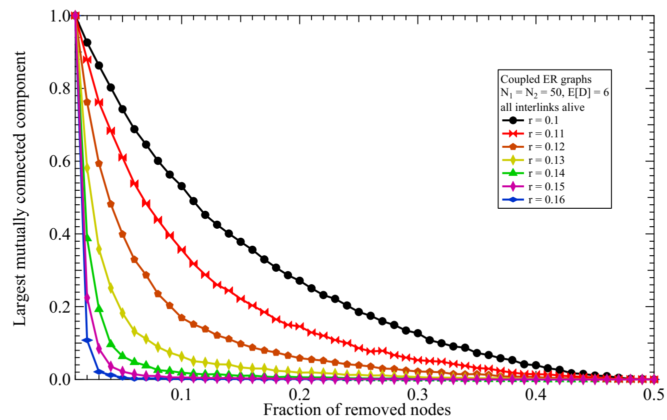

FIG. 13. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Erdős-Rényi graphs $G_p(N)$ with $N = 50$ and the average degree $E[D] = 6$. The radius r in the random geometric graph ranges from 0.1 to 0.16. The results are averaged over 10^3 realizations of interdependent graphs.

component decreases with the decrease of the radius r. For example, when a fraction 0.2 of nodes are removed, we have LMCC $= 0.79$ for $r = \sqrt{2}$ and LMCC $= 0.69$ for $r = 0.1$. The reason is that with the decrease of r, a node tends to have less interconnection nodes which increases the probability for a node to fail due to the failures of its interconnection nodes.

Figure 12 shows the largest mutually connected component as a function of the fraction of the removed nodes in coupled Barabási-Albert graphs. We assume a node alive when at least one of the interconnected nodes is alive. Coupled BA graphs have less distinguishable LMCC for different radius r compared to coupled ER graphs. The reason is twofold: (i) BA graphs are robust to random failures and (ii) when we increase the radius r, a node tends to have more than one interconnections.

Figure 13 shows the largest mutually connected component as a function of the fraction of the removed nodes in coupled Erdős-Rényi graphs. A node is alive when all of the interconnected nodes are alive. The LMCC in Figure 13 decreases dramatically fast with the increase of the fraction of removed nodes. With the increase of the radius r, LMCC decreases even faster. When $r = 0.2$, the failure of 2% of the nodes collapses the whole interdependent network.

Figure 14 shows the largest mutually connected component as a function of the fraction of the removed nodes in coupled Barabási-Albert graphs. A node is alive when all of the interconnected nodes are alive. For a small radius r, LMCC decreases slowly with the increase of the fraction of removed nodes because (i) BA graphs are robust to random failures and (ii) failures are less likely propagating to another network with small interconnections resulting from small r. However, for

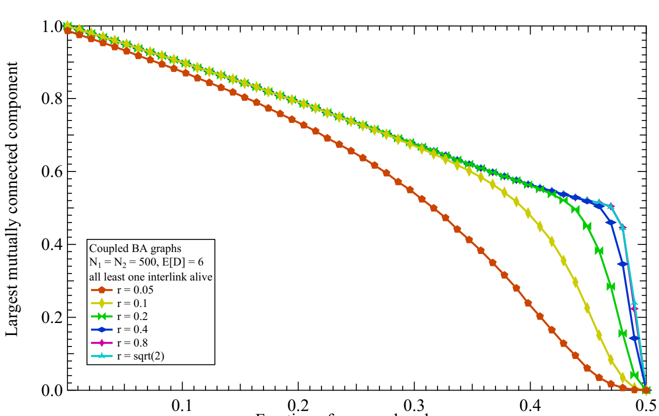

FIG. 12. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Barabási-Albert with $N = 500$ and the average degree $E[D] = 6$. The radius r in the random geometric graph is ranging from 0.05 to $\sqrt{2}$. The results are averaged over 10^3 realizations of interdependent graphs.

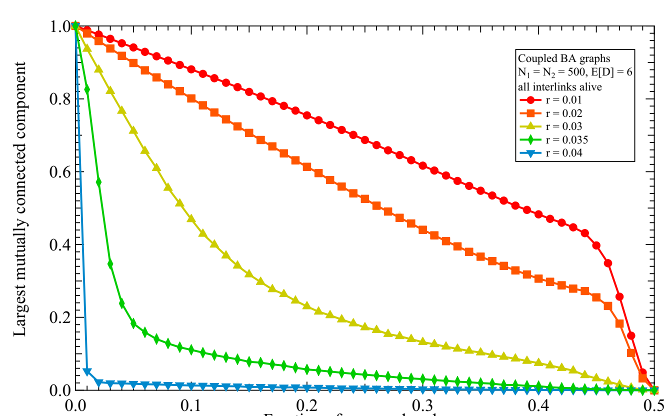

FIG. 14. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Barabási-Albert with $N = 500$ and the average degree $E[D] = 6$. The radius r in the random geometric graph ranges from 0.01 to 0.04. The results are averaged over 10^3 realizations of interdependent graphs.

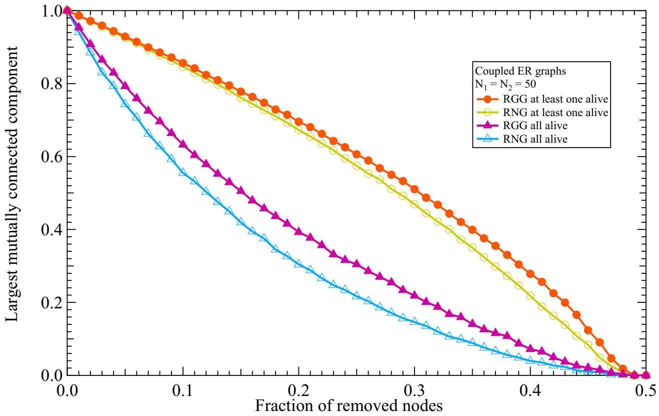

FIG. 15. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Erdős-Rényi graphs $G_p(N)$ with $N = 50$ and the average degree $E[D] = 6$. The interconnection topology is the relative neighborhood graph. The results are averaged over 10^3 realizations of interdependent graphs.

a larger radius r, LMCC decreases fast with the increase of removals $1 - q$.

B. Relative neighborhood graph as interconnection

To compare the interconnection structure of the relative neighborhood graph and the random geometric graph, we simulate the two topologies with the same interlink density derived in Theorems 1 and 2. Figures 15 and 16 show the largest mutually connected component as a function of the fraction $1 - q$ of the removed nodes in interdependent networks. The interdependent network consists of two Erdős-Rényi graphs with $N = 50$ and the average degree $E[D] = 6$ in Fig. 15 and consists of two Barabási-Albert graphs with $N = 500$ and the average degree $E[D] = 6$ in Fig. 16.

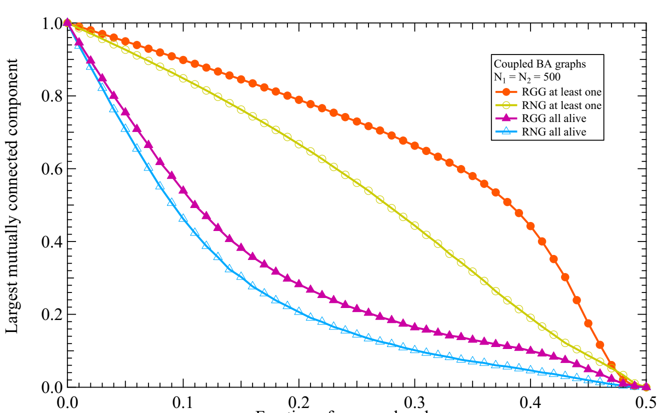

FIG. 16. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Barabási-Albert graphs with $N = 500$ and the average degree $E[D] = 6$. The interconnection topology is the relative neighborhood graph. The results are averaged over 10^3 realizations of interdependent graphs.

For both the assumptions of at least one interdependent node alive and all interdependent nodes alive, Fig. 15 shows that the interconnection structure of the random geometric graph is more robust compared to that of the relative neighborhood graph. An explanation is that interconnected links are evenly distributed in relative neighborhood graph, whereas in a random geometric graph, the interconnected links might be highly connect to few nodes depending on the location of nodes.

In Fig. 16, the interdependent graph with coupled BA graphs shows comparable results with coupled ER graphs. Random geometric graph performs much better than a relative neighborhood graph when at least one interlinks alive. For the assumption of all interlinks alive, a random geometric graph is also more robust than a relative neighborhood graph.

C. Real-world networks

To demonstrate the effectiveness of the two interconnection topologies, we interconnect two real-world coupled infrastructures in Italy [1,15] by the random geometric graph and the relative neighborhood graph and investigate their robustness under cascading failures.

One network is the Italian high-bandwidth backbone of the Internet consisting of $N = 39$ nodes and $L = 50$ links. The other network is the Italian high-voltage electrical transmission network consisting of $N = 310$ nodes and $L = 347$ links (excluding the double links). Given the geographical locations of the nodes in the Internet and in the electrical network, we generate interconnection topologies of the random geometric graph and the relative neighborhood graph as shown in Figs. 17 and 18.

Figure 19 shows the largest mutually connected component as a function of the fraction of removed nodes in coupled real-world networks. The interconnection topologies are the random geometric graph and the relative neighborhood graph with the same link density. For the assumption of at least one interlink alive, Figure 19 shows that the interconnection topology of the random geometric graph is more robust than that of the relative neighborhood graph. However, the relative neighborhood graph is more robust than the random geometric graph for the assumption of all interlinks alive.

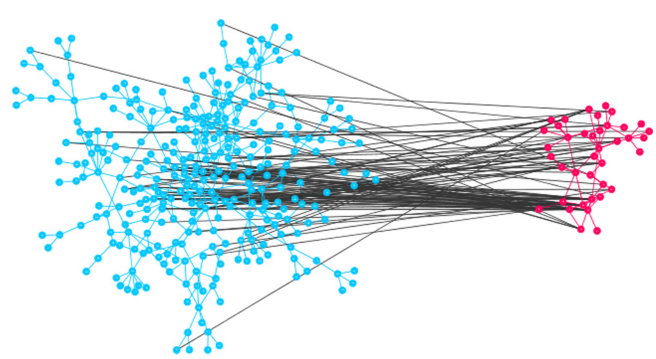

FIG. 17. Coupled Italian electrical transmission network (blue) and the Italian backbone of the Internet (red) with the interconnection topology of the random geometric graph.

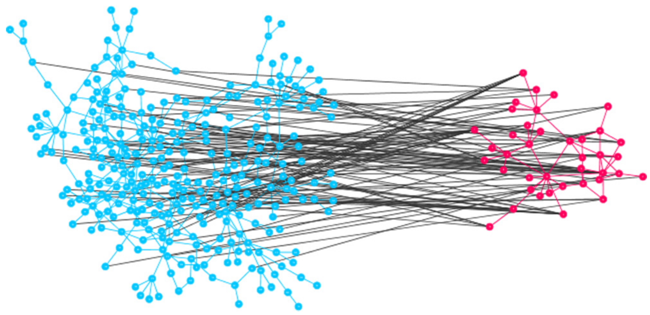

FIG. 18. Coupled Italian electrical transmission network (blue) and the Italian backbone of the Internet (red) with the interconnection topology of the relative neighborhood graph.

V. CONCLUSION

In this paper, we investigate two interconnection topologies for interdependent networks that incorporate the locations of nodes. The two topologies generalize the one-to-one interconnection to an arbitrary number of interconnections depending on the locations of nodes. We analyze the properties of the two topologies and the impact of the two interconnection structures on robustness of interdependent networks against cascading failures. Specifically, the derivation of the number of links in the two topologies enables the comparison of robustness performance between the two topologies. In particular, the random geometric graph provides the flexibility for network providers to determine the link density of interconnection in order to achieve the desired robustness level. The relative neighborhood graph, often used in wireless networks [16] to provide optimal coverage with least energy consumption, as an interconnection structure is less robust compared to the random geometric graph.

In addition, we propose the derivative of the largest mutually connected component as a new robust metric which addresses the impact of a small fraction of failed nodes. To avoid the collapse of the whole network, the proposed robustness metric quantifies the damage of networks triggered by a small fraction of failures, significantly smaller than the fraction at the critical threshold, that corresponds to the collapse of the whole network.

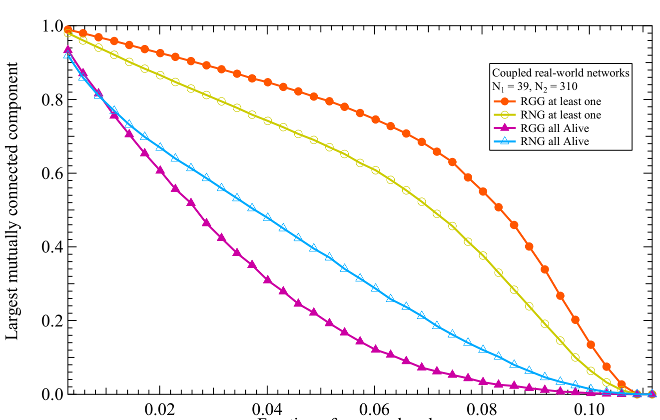

FIG. 19. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are the Italian high-bandwidth backbone of the Internet and the Italian high-voltage electrical transmission network. The interconnection topologies are the random geometric graph and the relative neighborhood graph with the same link density.

ACKNOWLEDGMENT

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APPENDIX A: ALGORITHMS: CASCADING AND COMPONENT

Algorithm 2 describes the function of cascading failures in interdependent networks. Lines 3 to 5 initialize a flag vector with flag $= 1$ if a node is not removed, otherwise flag $= 0$. Lines 6 to 9 remove the desired fraction $1 - q$ of nodes and set flag $= 0$ for removed nodes. Due to the interconnection structure, the initial failures cause dependent nodes to fail executed by lines 13 to 26. As specified in line 18, a node u in G_1 is removed if it does not belong to the largest mutually connected component C_{G_1} or it loses all the dependent nodes. The same rule is applied for a node in G_2 as shown in line 23. Lines 18 and 23 correspond to the scenario of at least one interdependent node alive. The failure of a node u may introduce further failures and may invoke a cascading failure (line 11 is true). The cascading process is terminated if no more nodes fail and delNodes (in line 12) is not changed. Line 28 returns the resulting graph after removing all the failed nodes.

Algorithm 3 extracts the largest mutually connected component from a given graph G. In line 3, we first obtain all the connected components C_i of G with sizes in descending order. Then lines 4 to 9 return the first connected component that includes nodes both in G_1 and G_2.

APPENDIX B: ALL INTERLINKS ARE ALIVE

Theorem 4. Consider an interdependent network consisting of two graphs G_1 and G_2. The interconnection topology between graphs G_1 and G_2 is the random geometric graph. Assume a node is alive when all of its interdependent nodes are alive. The fraction $S_i (i = 1,2)$ of the largest mutually connected component as a function of $1 - q$ removals is approximated by

$$S_1 = q\left[1 - \varphi_{C_{G_1}}(1)\right]\exp\left[-p_{ij}N\varphi_{C_{G_2}}(1)\right], \tag{B1}$$

$$S_2 = \left[1 - \varphi_{C_{G_2}}(1)\right]\exp\left\{p_{ij}N\left[q - q\varphi_{C_{G_1}}(1) - 1\right]\right\}, \tag{B2}$$

where

$$\varphi_{C_{G_1}}(1) = \varphi_{D_{G_1}}[1 - q\exp(-p_{ij}Nu_B)(1 - u_A)]$$
$$\varphi_{C_{G_2}}(1) = \varphi_{D_{G_2}}\{1 - \exp[p_{ij}N(q - qu_A - 1)](1 - u_B)\}$$

and

$$u_A = \varphi_{(D_{l^+}-1)}[1 - q\exp(-p_{ij}Nu_B)(1 - u_A)]$$
$$u_B = \varphi_{(D_{l^+}-1)}\{1 - \exp[p_{ij}N(q - qu_A - 1)](1 - u_B)\},$$

Algorithm 2. Function CASCADING($G, 1-q, \mathcal{N}_1, \mathcal{N}_2$)

```
1: Input: Graph G and fraction of removal 1 − q; Sets 𝒩1, 𝒩2 of nodes in G1 and G2, respectively
2: Output: endGraph: a graph after removing all the failed nodes from G
3: for each node u ∈ G do
4:   flag[u] ← 1
5: end for
6: for i = 1 to ⌈(1 − q)N⌉ do
7:   G ← G\{u1,u2, ··· ,ui} {ui is a randomly chosen node from graph G1}
8:   flag[u1,u2, ··· ,ui] ← 0;
9: end for
10: delNodes ← 1
11: while delNodes ≠ 0 do
12:   delNodes ← 0
13:   for each node u ∈ G do
14:     LMCC ← COMPONENT(G,𝒩1,𝒩2)
15:     C_G1 ← 𝒩1 ∩ LMCC
16:     C_G2 ← 𝒩2 ∩ LMCC
17:     N[u] ← get neighbors of u
18:     if u ∈ 𝒩1 and (u ∉ C_G1 or N[u] ∩ C_G2 = ∅) and flag[u]=1 then
19:       endGraph ← G\{u}
20:       flag[u] ← 0
21:       delNodes ← 1
22:       G ← endGraph
23:     else if u ∈ 𝒩2 and (u ∉ C_G2 or N[u] ∩ C_G1 = ∅) and flag[u]=1 then
24:       repeat lines 18–21
25:     end if
26:   end for
27: end while
28: return endGraph
```

where p_{ij} is the probability that there is a link l_{ij} between node i in graph G_1 and node j in graph G_2. $1-\varphi_{C_{G_1}}(1)$ is the fraction of nodes belonging to the giant component in graph G_1 and $1-\varphi_{C_{G_2}}(1)$ in graph G_2.

Proof. For a node n in G_1 with k dependent nodes in G_2, the probability that all the dependent nodes are alive follows

$$\sum_{k=0}^{\infty} \Pr[D_B = k](1-u_B)^k,$$

which can be written as the generating function $\varphi_{D_B}(1-u_B)$ of D_B with parameter $1-u_B$. Assuming D_B follows a binomial distribution, it holds [10] that $\varphi_{D_B}(1-u_B) = \exp(-E[D_B]u_B)$ for a large interconnection matrix B. When B is the random geometric graph, the degree distribution of D_B follows a binomial distribution [7] with average degree $E[D_B] = p_{ij}N$. Therefore, the probability that all the dependent nodes in G_2 of a node n in G_1 are alive is $\exp(-p_{ij}Nu_B)$.

The self-consistent equation for u_A in interdependent network with one-to-many interconnection follows

$$u_A = \varphi_{(D_{l^+}-1)}[1 - q\exp(-p_{ij}Nu_B)(1-u_A)],$$

where q is the probability for a node n to be occupied, and $\exp(-p_{ij}Nu_B)$ is the probability that all the interdependent nodes of a node n in G_1 belong to the giant component in graph G_2. Analogously,

$$u_B = \varphi_{(D_{l^+}-1)}\{1 - \exp[p_{ij}N(q - qu_A - 1)](1-u_B)\}.$$

Since we do not remove nodes from graph G_2 at the beginning, nodes in graph G_2 are occupied with probability 1. The

Algorithm 3. Function COMPONENT(G,$\mathcal{N}_1$,$\mathcal{N}_2$)

```
Input: Graph G; Sets 𝒩1, 𝒩2 of nodes in G1 and G2, respectively
Output: Largest mutually connected component LMCC
Get connected components C1, C2, ..., CN of G ordered as |C1| ⩽ ... ⩽ |CN|
for i = 1 to N do
  if Ci ∩ 𝒩1 ≠ ∅ and Ci ∩ 𝒩2 ≠ ∅ then
    LMCC ← Ci
    break
  end if
end for
return LMCC
```

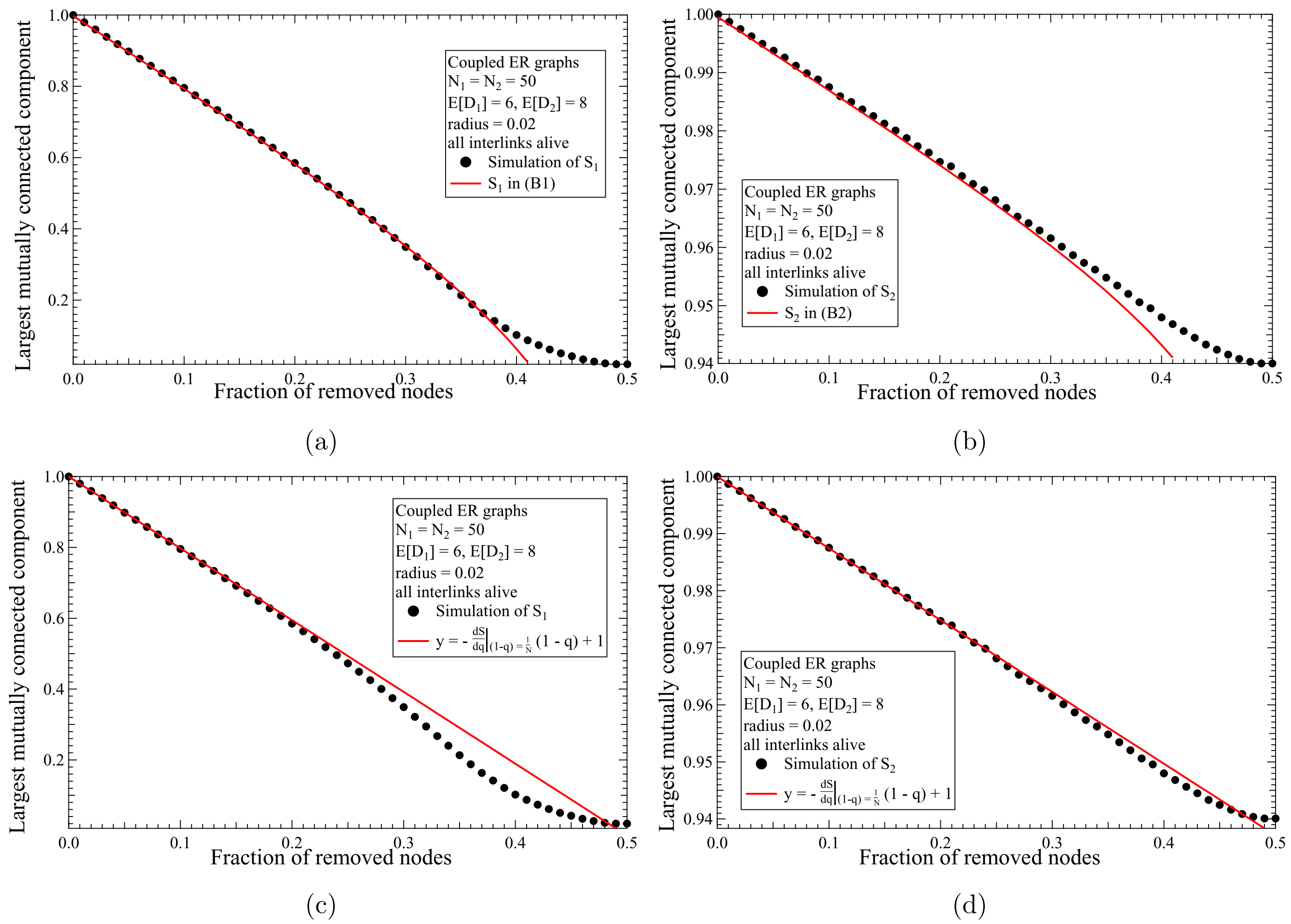

FIG. 20. Largest mutually connected component as a function of the fraction of removed nodes in interdependent networks. The coupled graphs are Erdős-Rényi graphs $G_p(N)$ with $N = 50$ and the average degrees $E[D_1] = 6$ and $E[D_2] = 8$. The interconnection topology is the random geometric graph with $r = 0.02$. The results are averaged over 10^4 realizations of interdependent graphs.

probability $\exp[p_{ij}N(q - qu_A - 1)]$ represents that all the dependent nodes of a node in G_2 are occupied and belong to the giant component in G_1. ■

For the scenario of all interdependent nodes alive, Figs. 20(a) and 20(b) show the simulation results and S_1 and S_2 in (B1) and (B2) in coupled ER graphs with interconnection of random geometric graph with radius $r = 0.02$. Figures 20(c) and 20(d) show the simulation results and a straight line $y = -\frac{dS_i}{dq}\big|_{1-q=\frac{1}{N}}(1-q) + 1 (i = 1,2)$, where the derivative $\frac{dS_i}{dq}(i = 1,2)$ is numerically computed based on (B1) and (B2). In Figs. 20(c) and 20(d), the straight line with slope $-\frac{dS_i}{dq}(i = 1,2)$ obtained from Theorem 4 shows a good approximation for the simulations for a small fraction of removals.

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