# Interconnectivity structure of a general interdependent network

P. Van Mieghem[*]

*Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands*

(Received 14 January 2016; published 7 April 2016)

A general two-layer network consists of two networks $G_1$ and $G_2$, whose interconnection pattern is specified by the interconnectivity matrix $B$. We deduce desirable properties of $B$ from a dynamic process point of view. Many dynamic processes are described by the Laplacian matrix $Q$. A regular topological structure of the interconnectivity matrix $B$ (constant row and column sum) enables the computation of a nontrivial eigenmode (eigenvector and eigenvalue) of $Q$. The latter eigenmode is independent from $G_1$ and $G_2$. Such a regularity in $B$, associated to equitable partitions, suggests design rules for the construction of interconnected networks and is deemed crucial for the interconnected network to show intriguing behavior, as discovered earlier for the special case where $B = wI$ refers to an individual node to node interconnection with interconnection strength $w$. Extensions to a general $m$-layer network are also discussed.

DOI: 10.1103/PhysRevE.93.042305

## I. INTRODUCTION

An interdependent network, also called an interconnected or multilayer network or network of networks, is a network consisting of different types of networks, which depend upon each other for their functioning [1]. For example, a power grid is steered by a computer network, which in turn needs electricity to function. Interconnected networks have been brought to the scientific scene by Buldyrev *et al.* [2], who illustrated the existence of dramatic cascading effects that could not occur in single networks. Since their pioneering work, a wealth of papers (see Ref. [3] and references therein) have appeared on interconnected networks. Different terminologies and mathematical representations, based on tensors, are discussed in Ref. [4]. The present work is motivated by the difficulty to specify the interconnection pattern between the layered or separate networks of an interdependent network. Particularly, the functional brain networks, measured with magnetoencephalography (MEG) at different frequencies, present slightly different properties of the anatomic brain and the challenge is to understand how these layered networks at different frequencies are interconnected to represent the total brain functioning (see, e.g., Refs. [5,6]).

The dynamics of many processes on networks can be described in terms of the Laplacian of the underlying graph topology, such as diffusion and approximate synchronization [4], and recently the (exact) prevalence in susceptible-infected-susceptible (SIS) epidemics on networks [7], analyzed in Ref. [8]. Radicchi [9] motivates the use of the Laplacian with many more examples. We study the eigenstructure of the Laplacian of a two-layered interconnected network (generalized to an $m$-layered network in Appendix B) as a function of the interconnectivity matrix. An insight presented here is that regularity in the interconnection pattern features attractive properties, that provide engineers with handles to control or uncouple the network's dynamics by changing the strength of the interconnectivity as well as by balancing or distributing that total strength over several interlinks, that connect nodes in different networks or layers. Previously reported special properties, for example a modular interconnected structure as in Ref. [10], can be related to regularity in the interconnection pattern. In fact, we show that most special phenomena observed—via the Laplacian spectrum—in the dynamic behavior of interconnected networks can be traced back to the graph theoretic concept of "equitable partitions" (explained in Sec. III A 2 and Appendix B).

After a definition of the interconnected network in Sec. II, the Laplacian eigensystem is studied. We derive a general quadratic form from which an interconnectivity energy is deduced. Although finding a nontrivial eigenmode (eigenvalue and eigenvector) is generally difficult to find, we show that, when the interconnection structure exhibits regularity, an interesting eigenmode can be determined in general, whose properties and consequences are studied in the remainder (Sec. III) of this paper. We conclude in Sec. IV. The proofs of the theorems are deferred to Appendix A.

## II. TWOFOLD INTERCONNECTED NETWORK

Consider an interconnected network $G$ with adjacency matrix

$$A = \begin{bmatrix} (A_1)_{n\times n} & B_{n\times m} \\ (B^T)_{m\times n} & (A_2)_{m\times m} \end{bmatrix}, \tag{1}$$

where $A_1$ is the $n \times n$ adjacency matrix of the graph $G_1$ with $n$ nodes, $A_2$ is the $m \times m$ adjacency matrix of the graph $G_2$ with $m$ nodes and $B$ is the $n \times m$ matrix interconnecting $G_1$ and $G_2$. The total number of nodes in $G$ is $N = n + m$. In the theory of interconnected or interdependent network (see Refs. [3,11,12] ), it is convenient to consider the interconnection matrix $B$ as a weighted matrix, whose elements are real, non-negative numbers, rather than just zero-one as in the square adjacency matrices $A_1$ and $A_2$. One of the main reasons for a real interconnection matrix $B$ is that the networks $G_1$ and $G_2$ are usually of a different type, e.g., a communication network $G_1$ that controls a power grid $G_2$. When $B = O$, the network $G$ is not interconnected anymore and falls apart into two separate networks $G_1$ and $G_2$. Thus, in the sequel, we assume that $B \neq O$, implying that at least one element $B_{ij} > 0$. We remark

[*]P.F.A.VanMieghem@tudelft.nl

 

that, possibly after a node relabeling, the adjacency matrix of any graph can be written as a block matrix (1), where two subgraphs $G_1$ and $G_2$ are interconnected by an $n \times m$ zero-one matrix $B$. The adjacency matrix $A$ in (1) represents a twofold interconnected network in the most general way. Two special cases of the interconnection matrix $B$, namely $B = wI$ and $B = wJ$, where $J$ is the all-one matrix, are briefly analyzed in Sec. III B.

Here, we follow the notation of my book [13]. The Laplacian $Q$ of $G$, corresponding to (1), equals

$$Q = \begin{bmatrix} (Q_1)_{n\times n}+\text{diag}[(Bu_m)_i] & -B_{n\times m} \\ -(B^T)_{m\times n} & (Q_2)_{m\times m}+\text{diag}\big[\big(B^T u_n\big)_i\big] \end{bmatrix}, \tag{2}$$

where $Q_1 = \Delta_1 - A_1$, $Q_2 = \Delta_2 - A_2$, and[1] $\Delta_k =$ diag $[(A_k u_k)_i] = \text{diag}[d_i(G_k)]$ for $k = 1,2$ and where $d_i(G_k)$ denotes the degree of node $i$ in the graph $G_k$. The all-one vector with $n$ components is denoted by $u_n$ and the subscript is omitted when the dimension is clear. We call the matrix $(Q_1)_{n\times n} +$ diag$[(Bu_m)_i]$ (and similarly for $G_2$) a generalized Laplacian, whose properties are studied in Refs. [14] and [1]. Only if $B$ is a zero-one matrix, the total number of links in $G$ equals

$$L = L_{G_2} + L_{G_1} + u_n^T B_{n\times m} u_m,$$

where $L_{G_k} = \frac{1}{2} u^T A_k u$ is the number of links in $G_k$.

Any $N \times N$ Laplacian matrix $Q$ is symmetric and positive semidefinite [13] and has a zero row and column sum, which is equivalent to the eigenvalue equation

$$Qu = 0$$

indicating that the smallest eigenvalue is $\mu_N = 0$, belonging to the eigenvector $x_N = u$.

## III. LAPLACIAN EIGENVALUE EQUATION

We order the eigenvalues of the $N \times N$ Laplacian $Q$ as $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{N-1} \geqslant \mu_N = 0$ and denote the eigenvector corresponding to the $k$-largest eigenvalue by $x_k$. The $k$th fundamental weight [15] of any Laplacian equals $w_k = u^T x_k = 0$, unless $k = N$, because eigenvectors are orthogonal, $x_k^T x_m = \delta_{mk}$. We write an $N \times 1$ vector $y$ as a block vector $y = (y_1^T, y_2^T)^T$, where $y_1$ is an $n \times 1$ and $y_2$ is an $m \times 1$ vector corresponding to the block structure of $A$ in (1). Hence, any eigenvector $x_k = x = (x_1^T, x_2^T)^T$ of the Laplacian $Q$ of the interconnected network $G$ with $1 \leqslant k < N$ obeys

$$u_n^T x_1 + u_m^T x_2 = 0 \tag{3}$$

while the normalization $x^T x = 1$ of the eigenvector $x$ translates to

$$x_1^T x_1 + x_2^T x_2 = 1. \tag{4}$$

[1] The diagonal matrix diag$(a_i)$, specified by the $i$th diagonal element, has the elements $a_1, a_2, \ldots, a_n$ on the diagonal.

The Laplacian eigenvalue equation for the eigenvector $x = (x_1^T, x_2^T)^T$ belonging to the eigenvalue $\mu$,

$$\begin{bmatrix} (Q_1)_{n\times n}+\text{diag}[(Bu_m)_i] & -B_{n\times m} \\ -(B^T)_{m\times n} & (Q_2)_{m\times m}+\text{diag}\big[\big(B^T u_n\big)_i\big] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is equivalent to the set

$$\begin{aligned} Q_1 x_1 + \text{diag}[(Bu_m)_i] x_1 - B x_2 &= \mu x_1 \\ Q_2 x_2 + \text{diag}[(B^T u_n)_i] x_2 - B^T x_1 &= \mu x_2. \end{aligned} \tag{5}$$

The quadratic form of $Q$ has the following property.

*Theorem 1.* Let $y = (y_1^T, y_2^T)^T$ be any real vector, then the quadratic form $y^T Q y$, where the Laplacian matrix $Q$ is defined in (2), equals

$$y^T Q y = y_1^T Q_1 y_1 + y_2^T Q_2 y_2 + R_{(y_1,y_2)}, \tag{6}$$

where

$$R_{(y_1,y_2)} = \sum_{i=1}^{n} \sum_{j=1}^{m} B_{ij} ((y_1)_i - (y_2)_j)^2, \tag{7}$$

which is always non-negative because $B_{ij} \geqslant 0$.

Since any Laplacian is positive semidefinite, each term in the Laplacian quadratic form (6) is non-negative. Due to our assumption $B \neq O$, (7) shows that $R_{(y_1,y_2)} = 0$ only if $(y_1)_i = (y_2)_j$ for all possible pairs $(i,j)$ of nodal interconnections with positive coupling strength $B_{ij} > 0$. In particular, when $y = u = (u_n^T, u_m^T)^T$, then $R_{(u_n,u_m)} = 0$ independently of the structure of $B$ [as also follows from (6) because $x_N = u$ is the eigenvector belonging to the zero Laplacian eigenvalue $\mu_N = 0$]. As a consequence of $R_{(y_1,y_2)} \geqslant 0$, we find with $y = x$ in (6) that any eigenvalue $\mu$ of $Q$ belonging to eigenvector $x = (x_1^T, x_2^T)^T$ is lower bounded by

$$\mu \geqslant x_1^T Q_1 x_1 + x_2^T Q_2 x_2.$$

Theorem 6 in Appendix B generalizes Theorem 1 to an $m$-fold interconnected network with $m \geqslant 2$.

We may interpret $R_{(y_1,y_2)}$ in (7) as the total interconnection energy between $G_1$ and $G_2$ due to the vector $y = (y_1^T, y_2^T)^T$. In such an interpretation, $y^T Q y$ represents the total network energy for a state vector $y$. Let us denote the link $l = (l^+, l^-)$, where the nodes $l^+$ and $l^-$ are the endpoints of the link and by $\mathcal{L}$ the set of all links in $G$. Then, a basic property (see, e.g., Ref. [13]) of the (unweighted) Laplacian $Q$ is $y^T Q y = \sum_{l\in\mathcal{L}} (y_{l^+} - y_{l^-})^2$ and, in the same vein, $R_{(y_1,y_2)}$ in (7) can be written as

$$R_{(y_1,y_2)} = \sum_{l: l^+ \in G_1 \text{ and } l^- \in G_2} B_{l^+ l^-} \big[(y_1)_{l^+} - (y_2)_{l^-}\big]^2.$$

Apart from the trivial Laplacian eigenvector $x_N = u$, finding nontrivial eigenvectors of $Q$ in (2) is generally difficult, even if eigenvectors of the networks $G_1$ and $G_2$ are known. A negative result is Theorem 2.

*Theorem 2.* The vector $y = (u_n^T, 0)^T$ nor $y = (0, u_m^T)^T$ can be an eigenvector of $Q$.

### A. Nontrivial eigenvector and eigenvalue solution

If the interconnected network $G$ with $N = m + n$ nodes has a special regular structure, we can determine at least two eigenmodes[2] of $Q$:

*Theorem 3.* Only if the $n \times m$ interconnection matrix $B$ has a constant row sum equal to $\frac{\mu^*}{N}m$ and a constant column sum equal to $\frac{\mu^*}{N}n$, which we call the regularity condition for $B_{n\times m}$,

$$\begin{cases} Bu_m = \mu^* \frac{m}{m+n} u_n \\ B^T u_n = \mu^* \frac{n}{m+n} u_m \end{cases} \tag{8}$$

then is

$$x = \frac{1}{\sqrt{N}} \left[ \sqrt{\tfrac{m}{n}} u_n^T \quad -\sqrt{\tfrac{n}{m}} u_m^T \right]^T \tag{9}$$

an eigenvector of $Q$, defined in (2), belonging to the eigenvalue

$$\mu^* = \left( \frac{1}{n} + \frac{1}{m} \right) u_n^T B_{n\times m} u_m \tag{10}$$

and $u_n^T B_{n\times m} u_m = \sum_{i=1}^n \sum_{j=1}^m B_{ij}$ equals the sum of the elements in $B$, specifying to the total strength of the interconnection between $G_1$ and $G_2$.

Since each element $B_{ij} \geqslant 0$, the eigenvalue $\mu^*$ in (10) can only be zero if $B = O$, in which case the two networks $G_1$ and $G_2$ are disconnected. With $N = m + n$, we can express this eigenvalue as

$$\mu^* = N \frac{u_n^T B_{n\times m} u_m}{nm},$$

where $\frac{u_n^T B_{n\times m} u_m}{nm} = \frac{\mu^*}{N}$ is the average coupling strength per element in $B$ and $\frac{\mu^*}{N}m = \frac{u_n^T B_{n\times m} u_m}{n}$ is the nodal coupling strength of each node $1 \leqslant i \leqslant n$ in $G_1$ to nodes in $G_2$ and similarly, $\frac{\mu^*}{N}n = \frac{u_n^T B_{n\times m} u_m}{m}$ is the nodal coupling strength of each node $1 \leqslant j \leqslant m$ in $G_2$ to nodes in $G_1$. Finally, we may regard $u_n^T B_{n\times m} u_m$ as the total coupling strength between the constituent network parts $G_1$ and $G_2$ to form $G$.

The eigenvector and eigenvalue in Theorem 3 are only determined by the interconnection matrix $B$ and are independent of the structure of $G_1$ and of $G_2$, because each eigenvector component of $x$ in (9) satisfies $Q_1 x_1 = Q_1 u_n = 0$ and, similarly, $Q_2 x_2 = Q_2 u_m = 0$. If we can control the coupling strength between $G_1$ and $G_2$, the eigenvalue $\mu^*$ in (10) can be changed at will. This independence of the network $G_1$ and $G_2$ property, earlier exploited in Ref. [3] to compute a particular interconnection strength separating the dynamics in $G_1$ and $G_2$, forms the major characteristic feature of a regular interlink interconnected network structure. If there is no special structure in the interconnection matrix $B$, then we may question whether there is a reason to study a block adjacency matrix of the form (1) instead of a single matrix $A$.

If $B$ is a zero-one matrix, then the condition in Theorem 3 shows that the interlink degree between a node $i \in G_1$ and a node $j \in G_2$ is constant. In other words, any node in $G_1$ is connected to a same number of nodes in $G_2$ (and vice versa). If both $G_1$ and $G_2$ have the same number of nodes, $n = m$, and for the particular form $B = wI_n$, the eigenvector $x$ in (9) exists and we find that $\mu^* = 2w$ (as in Ref. [3]). This particular case is discussed below in Sec. III B 1.

#### 1. Regularity and eigenstructure

A consequence of Theorem 3 is Corollary 1.

*Corollary 1.* If the regularity condition (8) on $B$ in Theorem 3 holds, then the $N \times N$ Laplacian matrix (2) simplifies to

$$Q = \begin{bmatrix} (Q_1)_{n\times n} + \mu^* \frac{m}{N} I_n & -B_{n\times m} \\ -(B^T)_{m\times n} & (Q_2)_{m\times m} + \mu^* \frac{n}{N} I_m \end{bmatrix} \tag{11}$$

and any other eigenvector $x = (x_1^T, x_2^T)^T$ of $Q$, apart from $x_N = u$ and $x = \frac{1}{\sqrt{N}}[\sqrt{\frac{m}{n}} u_n^T \; -\sqrt{\frac{n}{m}} u_m^T]^T$, must obey both $u_n^T x_1 = 0$ *and* $u_m^T x_2 = 0$.

If $B$ does not satisfy the regularity condition (8), then $x$ in (9) cannot be an eigenvector of $B$, implying that the eigenvectors $x = (x_1^T, x_2^T)^T$ of $Q$ belonging to $\mu$ cannot satisfy $u_n^T x_1 = 0$ *and* $u_m^T x_2 = 0$. Thus, the eigenvector property $u_n^T x_1 = 0$ *and* $u_m^T x_2 = 0$ is a fingerprint of the regularity of $B$ and reflects a notion of uncoupling of Laplacian eigenmodes of $G$ into those of both $G_1$ and $G_2$.

An interesting implication of Corollary 1 is that any eigenvector $x_1$ of $Q_1$ belonging to a positive eigenvalue also satisfies $u_n^T x_1 = 0$ (and similar for $Q_2$). Hence, we may ask whether $y = (x_1^T, x_2^T)^T$ is an eigenvector of $Q$.

*Theorem 4.* If the following conditions hold:

(i) $B$ satisfies the regularity condition (8) in Theorem 3;

(ii) $x_1$ is an eigenvector of both $Q_1$ and $BB^T$;

(iii) $x_2$ is an eigenvector of both $Q_2$ and $B^T B$;

(iv) the vectors $Bx_2$ and $x_1$ are parallel, i.e., $Bx_2 = [\mu(Q_1) + \frac{\mu^*}{N}m - \xi]x_1$;

(v) the vectors $B^T x_1$ and $x_2$ are parallel, i.e., $B^T x_1 = [\mu(Q_2) + \frac{\mu^*}{N}n - \xi]x_2$;

then $x = (x_1^T, x_2^T)^T$ an eigenvector of $Q$ in (11) belonging to the eigenvalue $\xi$ equal to

$$\xi = \frac{(\mu(Q_1) + \mu(Q_2) + \mu^*) x_1^T x_1 - \left(\mu(Q_2) + \frac{\mu^*}{N} n\right)}{2x_1^T x_1 - 1}, \tag{12}$$

which satisfies either

$$\xi \geqslant \max\left(\mu(Q_1) + \frac{\mu^*}{N}m, \mu(Q_2) + \frac{\mu^*}{N}n\right) \text{ or}$$
$$\xi \leqslant \min\left(\mu(Q_1) + \frac{\mu^*}{N}m, \mu(Q_2) + \frac{\mu^*}{N}n\right). \tag{13}$$

Theorem 4 contains many conditions, which make it hard for $y = (x_1^T, x_2^T)^T$ to be an eigenvector of $Q$. If $Q_1$ and $BB^T$ commute, then all eigenvectors of $Q_1$ and $BB^T$ are the same [13], but commutativity is an even harder confining condition. Theorem 4 thus illustrates that eigenmodes of the (Laplacian) dynamics of the constituent networks $G_1$ and $G_2$ can be directly reflected by the interconnected network $G$ under the rather stringent requirement and that, apart from the regularity of $B$, the conditions (iv) and (v) asks for an alignment of eigenmodes in $G_1$ and $G_2$. If $G_1$ and $G_2$ are different networks, such an alignment is likely to occur with low probability and the eigenmodes of $G$ are expected to show

[2] An eigenmode consists of an eigenvector and its corresponding eigenvalue.

a different, unrelated behavior. In other words, special design or special conditions (in a physical system) are needed to observe the peculiar transition in the coupling strength or normalized coupling strength $\mu^*$ in (10), reported earlier in Refs. [12,16] and further studied in Refs. [3,9].

#### 2. Regularity and equitable partitions

Any graph $G$ can be partitioned into nonempty subsets of nodes. The $N \times 2$ community matrix $S$ for a two-layer interdependent network is

$$S = \begin{bmatrix} u_n & 0 \\ 0 & u_m \end{bmatrix},$$

where $S_{ij}$ assigns node $i$ to the community or layer $j$. If the partition $\pi$ is equitable (or regular) [13, pp. 22–24], the quotient matrix $Q^\pi$ corresponding to the Laplacian $Q$ in (11) is

$$Q^\pi = \frac{\mu^*}{N}\begin{bmatrix} m & -m \\ -n & n \end{bmatrix}$$

from which we verify that $QS = SQ^\pi$. The eigenvalues of $Q^\pi$ are 0 and $\mu^*$, the latter belonging to the eigenvector $x^\pi = \frac{n}{\sqrt{n^2+m^2}}(1, -\frac{m}{n})$. If $\mu^\pi$ is an eigenvalue of $Q^\pi$ belonging to the eigenvector $x^\pi$ so that $\mu^\pi x^\pi = Q^\pi x^\pi$, then, after left multiplication with $S$ and invoking $QS = SQ^\pi$, we observe that

$$\mu^\pi S x^\pi = SQ^\pi x^\pi = QSx^\pi,$$

indicating that $Sx^\pi = \frac{1}{\sqrt{N}}[\sqrt{\frac{m}{n}}u_n^T \quad -\sqrt{\frac{n}{m}}u_m^T]^T$ is an eigenvector of $Q$ belonging to $\mu^\pi = \mu^*$ and illustrating that Theorem 3 is actually a consequence of the general theory of equitable partitions.

#### 3. Regularity and optimality

The eigenvalue $\mu^*$ in (10) as well as the interconnection energy (7) increases with any element $B_{ij} \geqslant 0$ of the interconnection matrix $B$. Thus, limiting the increase in the elements of $B$ seems a natural consequence of the analysis. Shakeri *et al.* [17] maximize the algebraic connectivity $\mu_{N-1}$, subject to a constant total interconnection strength $u_n^T B_{n\times m} u_m = c$, which is equivalent to a given $\mu^*$ in (10). In particular, Shakeri *et al.* [17] consider a special type of interconnected network $G_1$ and $G_2$ with equal number of nodes, thus $n = m$, and where the interconnection matrix $B = \mathrm{diag}(w_j)$ and $w_j \geqslant 0$ is the interconnection weight or strength between the node $j$ in $G_1$ and its peer (also labeled by $j$) in $G_2$. Their Laplacian is

$$Q_S = \begin{bmatrix} (Q_1)_{n\times n} + \mathrm{diag}(w_j) & -\mathrm{diag}(w_j) \\ -\mathrm{diag}(w_j) & (Q_2)_{n\times n} + \mathrm{diag}(w_j) \end{bmatrix}.$$

The vector $w = (w_1, w_2, \ldots, w_n)$ is a non-negative interconnection vector (which they denote by $w \geqslant 0$), on which they impose a total weight condition, namely $w^T u = c$. Their main result is

$$\max_{\substack{w\geqslant 0 \\ w^T u = c}} \mu_{N-1}(Q_S) \text{ is attained for } w = \frac{c}{n} u_n,$$

demonstrating that each interlink weight $w_k = \frac{c}{n} = \frac{2}{N} u^T B u = \frac{\mu^*}{2}$ is precisely equal, which emphasizes the extremality of the regularity of $B$. We add another extremal property of the nontrivial eigenmode when $B$ is regular:

*Theorem 5.* Let $B$ satisfy the regularity condition (8) in Theorem 3. Among all real vectors $y = (y_1^T, y_2^T)^T$ satisfying the normalization $y^T y = 1$, the Laplacian eigenvector $\frac{1}{\sqrt{N}}[\sqrt{\frac{m}{n}}u_n^T \quad -\sqrt{\frac{n}{m}}u_m^T]^T$ in (9) belonging to eigenvalue $\mu^*$ in (10) can attain the highest coupling energy $R_{(y_1,y_2)}$, defined in (7).

### B. Special interconnection matrices

We assume that $B$ has constant row and column sum, obeying the regularity condition (8).

#### 1. Individual node to node interconnection $B = wI$

The interconnection $B = wI$ has been investigated in depth in Refs. [3,9,12,16,17], where the number of nodes in both $G_1$ and $G_2$ is the same: $n = m$. Straightforward substitution of $B = wI$ in (10) shows that $\mu^* = 2w$ (as in Ref. [3], where $w$ was denoted by $p$). The total interconnection energy at eigenfrequency $\mu$ follows from (7) as

$$R_{(x_1,x_2)} = w\sum_{i=1}^{n}[(x_1)_i - (x_2)_i]^2 = w(x_1 - x_2)^T(x_1 - x_2).$$

In particular, at the eigenfrequency $\mu^* = 2w$ where $x_1 = \alpha u_n$ and $x_2 = \beta u_n$ (see Appendix A 3), the computations in Sec. III A result in

$$R_{\mu^*} = wn(\alpha-\beta)^2 = wn\alpha^2\left(1 + \frac{n}{m}\right)^2 = 2w = \mu^*.$$

We remark that, in particular, the conditions (iv) and (v) in Theorem 4 complicate the determination of additional eigenmodes of $Q$, even in this simple case with $B = wI$.

#### 2. All pair interconnection pattern $B = wJ$

If $B = wJ$, then

$$x_1^T J_{n\times m} x_2 = x_1^T (u_n u_m^T) x_2 = (u_n^T x_1)(u_m^T x_2).$$

With (3), we have

$$x_1^T J_{n\times m} x_2 = -(u_n^T x_1)^2.$$

Explicitly, when $x = x_N = u$, then $u^T J u = nm$ and for the nontrivial eigenvector $x$ in (9), we find that $x_1^T J_{n\times m} x_2 = -\frac{nm}{N}$, which is negative, while for any other Laplacian eigenmode, it holds that $x_1^T J_{n\times m} x_2 = 0$.

For any eigenvalue $\mu \neq \mu_N = 0$, (A1) indicates that

$$\begin{aligned} \mu(Q) &= x^T Q x = x_1^T Q_1 x_1 + x_2^T Q_2 x_2 \\ &\quad + w\{m x_1^T x_1 + n x_2^T x_2 - 2(u^T x_1)^2\}. \end{aligned}$$

The nontrivial eigenvector $x$ in (9) obeys the regularity condition (8)

$$\begin{cases} J_{n\times m} u_m = m u_n = \xi \frac{m}{m+n} u_n \\ J_{m\times n} u_n = n u_m = \xi \frac{n}{m+n} u_m, \end{cases}$$

which are eigenvalue equations, so that the eigenvalue $\mu^* = w\xi = w(n+m) = wN$, which agrees with (10).

The complement $Q^c$ possesses [13] the same eigenvectors as $Q$. In case $B = J$ (thus for $w = 1$), the complement $A^c$ of $A$ reduces to two not connected graphs $G_1^c$ and $G_2^c$, and the corresponding Laplacian $Q^c$ consists of two separate eigensystems of $Q_1^c$ and $Q_2^c$, and both have the same eigenvectors as $Q_1$ and $Q_2$. Thus, for $B = J$, the knowledge of the normalized eigenvectors $\{v_k\}_{1\leqslant k\leqslant n}$ of $Q_1$ and $\{r_l\}_{1\leqslant l\leqslant m}$ of $Q_2$ is sufficient to construct the eigenvector $x_{kl} = [\alpha_k v_k^T \quad \beta_l r_l^T]^T$ of $Q$, where the scalars $\alpha_k$ and $\beta_l$ obey $\alpha_k^2 + \beta_l^2 = 1$ as required by the normalization in (4). The orthogonality of eigenvectors demands that $x_{kl}^T x_{k'l'} = \alpha_k\alpha_{k'} v_k^T v_{k'} + \beta_l\beta_{l'} r_l^T r_{l'} = 0$, implying that $k \neq k'$ and $l \neq l'$. From the quadratic form (A1), the eigenvalues (when at least one $x_1 \neq u$) are

$$\mu(Q) = x^T Q x = [\mu_k(Q_1) + wm]\alpha_k^2 + [\mu_l(Q_2) + wm]\beta_l^2.$$

However, in total, there are $nm$ possible ways of constructing an eigenvector $x_{kl}$, while there can be only $n + m = N$ such eigenvectors. But, (A4) requires that $\mu(Q_1) + wm = \xi$ and $\mu(Q_2) + wm = \xi$, implying either $\alpha_k = 1$ and $\beta_l = 0$ (and vice versa), leading to the $n - 1$ eigenvector forms $\{[v_k^T \quad 0]^T\}_{1\leqslant k\leqslant n-1}$ and $m - 1$ form $\{[0 \quad r_l^T]^T\}_{1\leqslant l\leqslant m-1}$, apart from $x_N = u$ (both $v_n = u_n$ and $r_m = u_m$) and $x$ in (9). This computation, in agreement with Theorem 2 and Theorem 4, again underlines the restrictive nature of (A4), that asks for alignment of eigenvectors of $Q_1$ and $Q_2$. Finally, as a side effect of concentrating on the Laplacian (and due to the commutativity of any Laplacian $Q$ and $J$), we find that the eigenmodes of a complete interconnection pattern $B = J$ seems closely related to no interconnection $B = O$, which may seem counterintuitive at first glance.

### C. Algebraic connectivity of *G*

The second smallest eigenvalue $\mu_{N-1}$ of the Laplacian $Q$ is called by Fiedler [18] the algebraic connectivity. The algebraic connectivity is the most studied eigenvalue of $Q$ due to its appearance in many phenomena [13]. We derive several bounds for the algebraic connectivity $\mu_{N-1}(Q)$ of the interconnected graph $G$.

#### 1. Consequence of Theorems 1 and 3

By choosing $y_1$ equal to the $k$th normalized eigenvector (i.e., $y_1^T y_1 = 1$, while $y^T y = y_1^T y_1 + y_2^T y_2$) of $Q_1$ and $y_2$ equal to $l$th normalized eigenvector of $Q_2$, the quadratic form (6) reads

$$y^T Q y = \mu_k(Q_1) + \mu_l(Q_2) + R_{(y_1,y_2)}.$$

In particular, confining to the algebraic connectivity where $y_1 = x_{n-1}$ and $y_2 = x_{m-1}$ are the eigenvectors belonging to the respective algebraic connectivity $\mu_{n-1}$ in $G_1$ and $\mu_{m-1}$ in $G_2$, leads to

$$y^T Q y = \mu_{n-1}(Q_1) + \mu_{m-1}(Q_2) + R_{(x_{n-1},x_{m-1})},$$

where $u_n^T x_{n-1} = 0$ and $u_m^T x_{m-1} = 0$ (since eigenvectors are orthogonal), so that $y^T u_N = 0$. In that case, the Rayleigh inequality $y^T Q y \geqslant \mu_{m+n-1}(Q) y^T y = 2\mu_{m+n-1}(Q)$ leads to the upper bound for the algebraic connectivity of $Q$,

$$\mu_{N-1}(Q) \leqslant \tfrac{1}{2}[\mu_{n-1}(Q_1) + \mu_{m-1}(Q_2) + R_{(x_{n-1},x_{m-1})}]. \quad (14)$$

Let us investigate the smallest, nonzero eigenvalue $\mu$, corresponding to any eigenvector $x = (x_1^T, x_2^T)^T$ of $Q$ obeying $u_n^T x_1 = u_m^T x_2 = 0$ in Corollary 1, a necessary condition for regularity of $B$. The Rayleigh inequality [13] demonstrates for $u_n^T x_1 = 0$ that

$$x_1^T Q_1 x_1 \geqslant \mu_{n-1}(Q_1) x_1^T x_1$$

and, similarly for $u_m^T x_2 = 0$,

$$x_2^T Q_2 x_2 \geqslant \mu_{m-1}(Q_2) x_2^T x_2$$

with equality only if $x_1$ and $x_2$ are the eigenvector of $Q_1$ and $Q_2$ belonging to the algebraic connectivity, eigenvalue $\mu_{n-1}(Q_1)$ and $\mu_{m-1}(Q_2)$, respectively. Complementary to the upper bound (14), the quadratic form (6) leads to the lower bound (15) for the algebraic connectivity of $G$ with regular interconnection matrix $B$,

$$\mu_{N-1}(Q) \geqslant \mu_{n-1}(Q_1) x_1^T x_1 + \mu_{m-1}(Q_2) x_2^T x_2 + R_{(x_1,x_2)} \geqslant 0 \quad (15)$$

with $0 < x_2^T x_2 = 1 - x_1^T x_1 < 1$. Combining the upper bound (14) and the lower bound (15) yields, for a regular interconnection matrix $B$,

$$\begin{aligned} &\frac{\mu_{n-1}(Q_1)}{\frac{1}{x_1^T x_1}} + \frac{\mu_{m-1}(Q_2)}{\frac{1}{1-x_1^T x_1}} + R_{(x_1,x_2)} \\ &\quad\leqslant \mu_{N-1}(Q) \leqslant \frac{\mu_{n-1}(Q_1)}{2} + \frac{\mu_{m-1}(Q_2)}{2} \\ &\quad\quad + \frac{R_{(x_{n-1},x_{m-1})}}{2}. \end{aligned} \quad (16)$$

Even when both $G_1$ and $G_2$ are disconnected ($\mu_{n-1}(Q_1) = \mu_{m-1}(Q_2) = 0$), a positive interconnection energy $R_{(x_1,x_2)} > 0$ results in a connected interdependent network $G$ (i.e., $\mu_{N-1}(Q) > 0$). We observe from (16) that, if $\mu_{n-1}(Q_1) = \mu_{m-1}(Q_2)$, then we obtain the curious inequality

$$\mu_{n-1}(Q_1) + R_{(x_1,x_2)} \leqslant \mu_{N-1}(Q) \leqslant \mu_{n-1}(Q_1) + \frac{R_{(x_{n-1},x_{m-1})}}{2},$$

illustrating that $0 \leqslant R_{(x_1,x_2)} \leqslant \mu_{N-1}(Q) - \mu_{n-1}(Q) \leqslant \frac{R_{(x_{n-1},x_{m-1})}}{2}$. The interconnection energy $R_{(x_1,x_2)}$ for the Fiedler eigenvector $x$ of $Q$ is smaller than half the interconnection energy $R_{(x_{n-1},x_{m-1})}$ of the individual Fiedler vectors $x_{n-1}$ of $Q_1$ and $x_{m-1}$ of $Q_2$, both belonging to a same algebraic connectivity $\mu_{n-1}(Q_1) = \mu_{m-1}(Q_2)$.

The scaling of elements in $B$ also causes that the eigenvalue $\mu^*$ in (10) is not necessarily equal to the second smallest eigenvalue $\mu_{N-1}$. Indeed, by lowering the total coupling strength $u_n^T B_{n\times m} u_m$, we can always force $\mu^*$ to be lower than $\mu_{N-1}(Q)$, because, from (15), we have that $\mu_{N-1}(Q) \geqslant \min_{(x_1^T x_1, x_2^T x_2)}\{\mu_{n-1}(Q_1) + \mu_{m-1}(Q_2)\} > 0$ for connected networks $G_1$ and $G_2$. The possibility of modifying the total coupling strength $u_n^T B_{n\times m} u_m$ leads to consequences elaborated in Ref. [3], where the coupling strength $w$ in $B = wI$ was computed so that $\mu^* = \mu_{N-1}(Q)$.

#### 2. Interlacing

The interlacing theorem for symmetric matrices [13] tells us that

$$\begin{cases} \mu_{m+i}(Q) \leqslant \mu_i\{Q_1 + \mathrm{diag}[(Bu_m)_q]\} \leqslant \mu_i(Q) & \text{for any} \quad 1 \leqslant i \leqslant n \\ \mu_{n+j}(Q) \leqslant \mu_j\{Q_2 + \mathrm{diag}[(B^T u_n)_i]\} \leqslant \mu_j(Q) & \text{for any} \quad 1 \leqslant j \leqslant m. \end{cases}$$

In particular, for the algebraic connectivity, the interlacing theorem states

$$\begin{cases} \mu_{m+n-1}(Q) \leqslant \mu_{n-1}\{Q_1 + \mathrm{diag}[(Bu_m)_q]\} \leqslant \mu_{n-1}(Q) \\ \mu_{n+m-1}(Q) \leqslant \mu_{m-1}\{Q_2 + \mathrm{diag}[(B^T u_n)_i]\} \leqslant \mu_{m-1}(Q) \end{cases}$$

so that the algebraic connectivity $\mu_{m+n-1}(Q) = \mu_{N-1}(Q)$ of the interconnected graph $G$ is upper bounded by

$$\mu_{N-1}(Q) \leqslant \min\,(\mu_{n-1}(Q_1 + \mathrm{diag}((Bu_m)_q)),$$
$$\mu_{m-1}(Q_2 + \mathrm{diag}((B^T u_n)_i))).$$

If $\mathrm{diag}((Bu_m)_q) = b_m I$, then $\mu_{n-1}(Q_1 + \mathrm{diag}((Bu_m)_q)) = b_m + \mu_{n-1}(Q_1)$ (and similar for the the graph $G_2$), so that

$$\mu_{N-1}(Q) \leqslant \min[b_m + \mu_{n-1}(Q_1), b_n + \mu_{m-1}(Q_1)].$$

If we assume in addition that $b_m = b_n = w$, then

$$\mu_{N-1}(Q) \leqslant w + \min[\mu_{n-1}(Q_1), \mu_{m-1}(Q_1)].$$

The regime in which $\mu_{N-1}(Q) \geqslant \max\,[\mu_{n-1}(Q_1), \mu_{m-1}(Q_1)]$ is related to superdiffusion and is possible for a certain interconnection strength $w$ (see, e.g., Ref. [3]).

## IV. CONCLUSION

As shown in Sec. III A, we believe that a regular interconnection matrix $B$ with constant row sum and column sum (based on the theory of equitable partitions) is adequate to engineer or approach interdependent networks. For a regular interconnection matrix $B$, there always exist a nontrivial eigenvector (Theorem 3), which only depends on $B$ and whose eigenvalue can be controlled via the interconnection strength (sum of all elements in $B$). This special eigenmode gives rise to remarkable physical properties as illustrated in Ref. [3] and references therein. The fact that the interconnection matrix $B$ is usually defined with real elements simplifies, besides the determination of the coupling strength between potentially different networks $G_1$ and $G_2$, also the flexibility to shape or control the topological structure of $B$: an arbitrary number of links from node $i \in G_1$ to $G_2$ can be used as long as the sum of their interconnection strength $\sum_{j=1}^{m} B_{ij}$ is constant for each node $i$ in $G_1$. The constraint to construct a matrix with a constant row sum (and column sum) is more realistic for a real matrix than for a matrix with zero-one elements. In the latter case, the constraints on $B$ would imply that each node $i$ in $G_1$ (and vice versa for $G_2$) has the same number of links to nodes in $G_2$, which in many real-world cases is not justifiable. However, even for a real interconnection matrix $B$, the regularity constraint means that each node $i$ in $G_1$ is coupled to nodes in $G_2$ with equal strength. If an equal strength coupling is not defendable, the study of the interconnected network $G$ can hardly benefit from the knowledge of its constituent parts $G_1$ and $G_2$, because, as shown here, the eigenmodes of $G$ are hardly related to those of $G_1$ and $G_2$. Thus, to some extent and from a graph theoretical point of view, such a network $G$ can better be analyzed as a single (though weighted) network.

## ACKNOWLEDGMENTS

We are very grateful to Faryad Darabi Sahneh and Prejaas Tewarie for their valuable comments.

## APPENDIX A: PROOFS

### 1. Proof of Theorem 1

*Proof.* The quadratic form $y^T Q y$ equals

$$\begin{aligned} y^T Q y &= \begin{bmatrix} y_1^T & y_2^T \end{bmatrix} \begin{bmatrix} (Q_1)_{n\times n} + \mathrm{diag}((Bu_m)_i) & -B_{n\times m} \\ -(B^T)_{m\times n} & (Q_2)_{m\times m} + \mathrm{diag}((B^T u_n)_i) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} y_1^T & y_2^T \end{bmatrix} \begin{bmatrix} ((Q_1)_{n\times n} + \mathrm{diag}((Bu_m)_i))y_1 - B_{n\times m} y_2 \\ ((Q_2)_{m\times m} + \mathrm{diag}((B^T u_n)_i))y_2 - (B^T)_{m\times n} y_1 \end{bmatrix} \\ &= y_1^T (Q_1)_{n\times n} y_1 + y_1^T \mathrm{diag}((Bu_m)_i) y_1 - y_1^T B_{n\times m} y_2 + y_2^T (Q_2)_{m\times m} y_2 + y_2^T \mathrm{diag}((B^T u_n)_i) y_2 - y_2^T (B^T)_{m\times n} y_1. \end{aligned}$$

Since $(y_1^T B y_2)^T = y_2^T B^T y_1$ is a scalar, whose transpose is equal to itself, $(y_1^T B y_2)^T = y_1^T B y_2$ and we arrive at

$$y^T Q y = y_1^T Q_1 y_1 + y_2^T Q_2 y_2 + y_1^T \mathrm{diag}((Bu_m)_i) y_1 + y_2^T \mathrm{diag}((B^T u_n)_i) y_2 - 2 y_1^T B_{n\times m} y_2, \tag{A1}$$

where

$$\begin{cases} y_1^T \mathrm{diag}((Bu_m)_i) y_1 = \sum_{r=1}^{n} \left( \sum_{j=1}^{m} B_{rj} \right) (y_1)_r^2 \\ y_2^T \mathrm{diag}((B^T u_n)_i) y_2 = \sum_{q=1}^{m} \left( \sum_{i=1}^{n} B_{iq} \right) (y_2)_q^2 \\ y_1^T B_{n\times m} y_2 = \sum_{i=1}^{n} \sum_{j=1}^{m} B_{ij} (y_1)_i (y_2)_j. \end{cases}$$

Hence, the interconnection part in (A1) equals

$$R_{(y_1,y_2)} = y_1^T \mathrm{diag}((Bu_m)_i)y_1 + y_2^T \mathrm{diag}((B^T u_n)_i)y_2 - 2y_1^T B_{n\times m} y_2 = \sum_{i=1}^n \sum_{j=1}^m B_{ij}\{(y_1)_i^2 + (y_2)_j^2 - 2(y_1)_i(y_2)_j\}$$

from which (7) follows. ■

### 2. Proof of Theorem 2

*Proof.* The eigenvalue equation in (5) reduces with $x_1 = y_1$ and $x_2 = 0$ to

$$\begin{cases} Q_1 y_1 + \mathrm{diag}[(Bu_m)_i]y_1 = \mu y_1 \\ B^T y_1 = 0, \end{cases}$$

where

$$0 = B^T y_1 = \sum_{i=1}^n B_{ik}(y_1)_i \quad \text{for } 1 \leqslant k \leqslant m,$$

while, for $1 \leqslant i \leqslant n$,

$$\mathrm{diag}[(Bu_m)_i]y_1 = (y_1)_i \sum_{j=1}^m B_{ij}.$$

The condition $(B^T)_{m\times n} y_1 = 0$ expresses that $y_1$ is orthogonal to each of the $m$ column vectors of $B_{n\times m}$. Since $B_{ij} \geqslant 0$ and $B \neq O$, the condition $B^T y_1 = 0$ shows that $y_1 \neq u_n$. ■

If $y_1 \neq u_n$, then the vector $y = (y_1^T, 0)^T$ can be an eigenvector of $Q$, for example when $B = J$ as shown in Sec. III B 2.

### 3. Proof of Theorem 3

*Proof.* If $x_1 = \alpha u_n$ and $x_2 = \beta u_m$ are parts of the eigenvector $x^T = (x_1^T, x_2^T)$ of $Q$ not equal to $x = u$, then the orthogonality condition (3) yields $\beta = -\alpha \frac{n}{m}$, while the normalization (4) leads to $\alpha^2 n + \beta^2 m = 1$. The latter is an ellipse in the variables $\alpha$ and $\beta$ around the origin with axis $\frac{1}{\sqrt{n}}$ and $\frac{1}{\sqrt{m}}$, while the former is a line through the origin. Their intersection results in two possible points $(\alpha,\beta)$ as in Ref. [19]:

$$\alpha = \pm\sqrt{\frac{m}{nN}} \quad \beta = \mp\sqrt{\frac{n}{mN}} \tag{A2}$$

but the two resulting eigenvectors are the same (apart from the factor $-1$). In order for $x^T = (\alpha u_n^T, \beta u_m^T)$, with $\alpha$ and $\beta$ defined in (A2), to be an eigenvector of $Q$ belonging to the eigenvalue $\mu^*$, $x$ must obey the eigenvalue equation (5)

$$\begin{cases} \alpha \mathrm{diag}[(Bu_m)_i]u_n - \beta B u_m = \mu^* \alpha u_n \\ \beta \mathrm{diag}[(B^T u_n)_i]u_m - \alpha B^T u_n = \mu^* \beta u_m. \end{cases}$$

Since $\mathrm{diag}[(Bu_m)_i]u_n = Bu_m$ and $\mathrm{diag}[(B^T u_n)_i]u_m = B^T u_n$, we have

$$\begin{cases} Bu_m = \mu^* \frac{\alpha}{\alpha-\beta} u_n \\ B^T u_n = -\mu^* \frac{\beta}{\alpha-\beta} u_m. \end{cases}$$

With $\frac{\alpha}{\alpha-\beta} = \frac{m}{m+n}$ and $\frac{\beta}{\alpha-\beta} = -\frac{n}{m+n}$, we arrive at the regularity condition (8), which means with $N = m+n$ that $B$ must have a constant row equal to $\mu^* \frac{m}{N}$ and column sum equal to $\mu^* \frac{n}{N}$. With these eigenvector parts $x_1 = \alpha u_n$ and $x_2 = \beta u_m$, the quadratic form (A1) becomes

$$\begin{aligned} \mu^* &= x^T Q x = \alpha^2 u_n^T \mathrm{diag}[(Bu_m)_i]u_n + \beta^2 u_m^T \mathrm{diag}[(B^T u_n)_i]u_m \\ &\quad - 2\alpha\beta u_n^T B_{n\times m} u_m \\ &= (\alpha-\beta)^2 u_n^T B_{n\times m} u_m. \end{aligned}$$

Finally, with $\beta = -\alpha \frac{n}{m}$ and $\alpha^2 = \frac{m}{nN}$, we arrive at (10). ■

### 4. Proof of Corollary 1

*Proof.* If Theorem 3 holds, it follows from (8) that

$$\mathrm{diag}[(Bu_m)_i] = \mu^* \frac{m}{N} I_n \text{ and } \mathrm{diag}[(B^T u_n)_i] = \mu^* \frac{n}{N} I_m$$

leading to (11). Any other eigenvector $x$ of $Q$ must be orthogonal to the above two eigenvectors $x_N = u$ and $\frac{1}{\sqrt{N}}[\sqrt{\frac{m}{n}} u_n^T \quad -\sqrt{\frac{n}{m}} u_m^T]^T$ and must obey

$$\begin{cases} u_n^T x_1 + u_m^T x_2 = 0 \\ m u_n^T x_1 - n u_m^T x_2 = 0 \end{cases} \Leftrightarrow \left\{ \begin{bmatrix} 1 & 1 \\ m & -n \end{bmatrix} \begin{bmatrix} u_n^T x_1 \\ u_m^T x_2 \end{bmatrix} = 0. \right. \tag{A3}$$

Since the determinant in (A3) is $-(m+n) = -N \neq 0$, any other eigenvector must obey both $u_n^T x_1 = 0$ *and* $u_m^T x_2 = 0$. ■

### 5. Proof of Theorem 4

*Proof.* The eigenvalue equation $Qx = \xi x$ for $Q$ in (11) becomes with (5)

$$\begin{cases} (Q_1)_{n\times n} x_1 + \mu^* \frac{m}{m+n} x_1 - B_{n\times m} x_2 = \xi x_1 \\ (Q_2)_{m\times m} x_2 + \mu^* \frac{n}{m+n} x_2 - (B^T)_{m\times n} x_1 = \xi x_2. \end{cases}$$

Since $x_1$ is an eigenvector $Q_1$, then $Q_1 x_1 = \mu(Q_1) x_1$ (and similar for $x_2$ and $Q_2$), we obtain

$$\begin{cases} B_{n\times m} x_2 = \left(\mu(Q_1) + \frac{\mu^*}{N} m - \xi\right) x_1 \\ (B^T)_{m\times n} x_1 = \left(\mu(Q_2) + \frac{\mu^*}{N} n - \xi\right) x_2. \end{cases} \tag{A4}$$

After left multiplying the first equation by $B^T$ and using the second in (A4), we find the eigenvalue equations

$$\begin{cases} B^T B x_2 = \zeta x_2 \\ B^T x_1 = \left(\mu(Q_2) + \frac{\mu^*}{N} n - \xi\right) x_2, \end{cases}$$

where

$$\zeta = \left(\mu(Q_1) + \frac{\mu^*}{N} m - \xi\right)\left(\mu(Q_2) + \frac{\mu^*}{N} n - \xi\right). \tag{A5}$$

Alternatively, left multiplying the second equation by $B$ and using the first in (A4) leads to

$$\begin{cases} B x_2 = \left(\mu(Q_1) + \frac{\mu^*}{N} m - \xi\right) x_1 \\ B B^T x_1 = \zeta x_1. \end{cases}$$

Hence, we have demonstrated the conditions in Theorem 4: $x_1$ is an eigenvector of both $Q_1$ and $BB^T$ and $x_2$ is an eigenvector of both $Q_2$ and $B^TB$, while, in addition, the vector $Bx_2$ and $x_1$, as well as $B^Tx_1$ and $x_2$, are parallel.

The remainder concentrates on the determination of the eigenvalue $\xi$. By left multiplying the first equation in (A4) by $x_1^T$ (and similar for the second equation), we obtain[3]

$$\frac{x_1^TBx_2}{x_1^Tx_1} = \mu(Q_1) + \frac{\mu^*}{N}m - \xi$$
$$\text{and}\quad \frac{x_2^TB^Tx_1}{x_2^Tx_2} = \mu(Q_2) + \frac{\mu^*}{N}n - \xi \tag{A6}$$

and since the scalar $x_1^TBx_2 = x_2^TB^Tx_1$, we find that[4]

$$\left(\mu(Q_1) + \frac{\mu^*}{N}m - \xi\right)x_1^Tx_1 = \left(\mu(Q_2) + \frac{\mu^*}{N}n - \xi\right)x_2^Tx_2.$$

The normalization (4) leads to the eigenvalue $\xi$ of $Q$ in (12). Furthermore, we see that $\zeta x_1^Tx_1 = x_1BB^Tx_1 = \|B^Tx_1\|_2^2$ and $\zeta x_2^Tx_2 = x_2B^TBx_2 = \|Bx_2\|_2^2$ so that[5]

$$\zeta = \|B^Tx_1\|_2^2 + \|Bx_2\|_2^2 \geqslant 0$$

and (A5) indicate that either $\mu(Q_1) + \mu^*\frac{m}{N} \leqslant \xi$ and $\mu(Q_2) + \mu^*\frac{n}{N} \leqslant \xi$ or $\mu(Q_1) + \mu^*\frac{m}{N} \geqslant \xi$ and $\mu(Q_2) + \mu^*\frac{n}{N} \geqslant \xi$. Combined, we find that the eigenvalue $\xi$ of $Q$ obeys (13). ■

#### 6. Proof of Theorem 5

*Proof.* If $B_{\max} = \max_{1\leqslant i\leqslant n; 1\leqslant j\leqslant m} B_{ij}$ is the maximum element of the interconnection matrix $B$, then the coupling energy $R_{(y_1,y_2)}$ for any real vector $y = (y_1^T, y_2^T)^T$ in (7) is upper bounded by

$$R_{(y_1,y_2)} \leqslant B_{\max}\sum_{i=1}^{n}\sum_{j=1}^{m}((y_1)_i - (y_2)_j)^2 = B_{\max}R_{(y_1,y_2)}|_{B=J},$$

which is rewritten as

$$\frac{R_{(y_1,y_2)}}{B_{\max}} \leqslant m\sum_{i=1}^{n}(y_1)_i^2 + n\sum_{j=1}^{m}(y_2)_j^2 - 2\sum_{i=1}^{n}(y_1)_i\sum_{j=1}^{m}(y_2)_j$$
$$= m\|y_1\|^2 + n\|y_2\|^2 - 2\left(u_n^Ty_1\right)\left(u_m^Ty_2\right).$$

Given the constraint on the norm, $\|y_1\|^2 + \|y_2\|^2 = \|y\|^2 = 1$ implying that $\|y_1\|^2 \leqslant 1$ and $\|y_2\|^2 \leqslant 1$, then

$$\frac{R_{(y_1,y_2)}}{B_{\max}} \leqslant N - 2\left(u_n^Ty_1\right)\left(u_m^Ty_2\right).$$

If either $y_1$ is orthogonal to $u_n$ (thus $u_n^Ty_1 = 0$) or $u_m^Ty_2 = 0$ as in any other eigenmode of $Q$ (Corollary 1), we observe that the right-hand side reduces to $\frac{R_{(y_1,y_2)}}{B_{\max}} \leqslant N$. On the other hand, if $y_1 = \alpha u_n$ and $y_2 = \beta u_m$ as in Theorem 3 where $\alpha\beta = -\frac{1}{N}$, the right-hand side is maximized,

$$\frac{R_{(\alpha u_n, Bu_m)}}{B_{\max}} \leqslant N + 2\frac{nm}{N} \leqslant \frac{3}{2}N$$

because[6] $\frac{2nm}{N} \leqslant \frac{N}{2}$. Hence, we conclude that, among all eigenmodes of $Q$ and, hence,[7] among all possible normalized vectors $y = (y_1^T, y_2^T)^T$, the eigenmode $x = \frac{1}{\sqrt{N}}[\sqrt{\frac{m}{n}}u_n^T \quad -\sqrt{\frac{n}{m}}u_m^T]^T$ belonging to the eigenvalue $\mu^*$ in Theorem 3 can obtain the highest possible coupling energy. ■

### APPENDIX B: *m*-FOLD INTERCONNECTED NETWORK

We extend the analysis to $m$ interconnected networks with adjacency matrix

$$A = \begin{bmatrix} A_1 & B_{12} & B_{13} & \cdots & B_{1m} \\ B_{21} & A_2 & B_{23} & \cdots & B_{2m} \\ B_{31} & B_{32} & A_3 & \cdots & B_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & B_{m3} & \cdots & A_m \end{bmatrix},$$

where $A_k$ is the adjacency matrix of the graph $G_k$ with $n_k$ nodes and $B_{ij}$ is the $n_i \times n_j$ matrix interconnecting $G_i$ and $G_j$. The total number of nodes in $G$ is $N = \sum_{k=1}^m n_k$. We confine ourselves to symmetric adjacency matrices $A = A^T$ as well as $A_k^T = A_k$ for any subgraph $G_k$ of $G$. One of the consequences of symmetry is that $(B_{ji})_{n_j\times n_i} = B_{ij}^T$ and $B_{jj} = O_{n_j\times n_j}$. The corresponding Laplacian $Q = \Delta - A$ of $G$ equals

$$Q = \begin{bmatrix} Q_1 + C_1 & -B_{12} & -B_{13} & \cdots & -B_{1m} \\ -B_{21} & Q_2 + C_2 & -B_{23} & \cdots & -B_{2m} \\ -B_{31} & -B_{32} & Q_3 + C_3 & \cdots & B_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B_{m1} & -B_{m2} & -B_{m3} & \cdots & Q_m + C_m \end{bmatrix}, \tag{B1}$$

where the diagonal matrix $C_k = \operatorname{diag}(\sum_{j=1}^m (B_{kj}u_{n_j})_q)$ and $u_{n_k}$ is the $n_k \times 1$ all-one vector. The Laplacian $Q_k = \Delta_k - A_k$ of subgraph $G_k$ has diagonal elements equal to the degree of its nodes, which is the number of intralinks incident to a node, without the interconnection links that are specified by the $C_k$ matrix.

---

[3] Since $u_n^Tx_1 = 0$ and $u_m^Tx_2 = 0$, it follows from (A4) that

$$\begin{cases} u_n^TB_{n\times m}x_2 = 0 \\ u_m^T(B^T)_{m\times n}x_1 = 0. \end{cases}$$

Thus, the scalar $u_n^TBx_2 = x_2^T(u_n^TB)^T = x_2^TB^Tu_n = 0$ (and similarly $u_m^TB^Tx_1 = x_1^TBu_m = 0$) and these scalar products are compatible with the regularity condition (8).

[4] Combining one of the equations in (A6) and the quadratic form (6) alternatively leads to (12). If $x_1^Tx_1 = \frac{1}{2} = x_2^Tx_2$, then it must hold that $\mu(Q_1) - \mu(Q_2) = \frac{\mu^*}{N}(n - m)$.

[5] Since $x_1^TBx_2 = x_2^TB^Tx_1$, a non-negative $\zeta$ also follows from (A6) after multiplying both equations.

[6] Combine $N^2 = (n + m)^2$ and $(m - n)^2 \geqslant 0$.

[7] Each $N \times 1$ vector $y$ can be written as a linear combination of eigenvectors of $Q$.

The block vector $y = (y_1^T, y_2^T, \dots, y_m^T)^T$ is normalized so that $y^T y = 1$ and

$$\sum_{j=1}^{m} y_j^T y_j = 1. \tag{B2}$$

*Theorem 6.* Let $y = (y_1^T, y_2^T, \dots, y_m^T)^T$ be any real vector, then the quadratic form $y^T Q y$ of the Laplacian (B1) of $m$ interconnected networks equals

$$y^T Q y = \sum_{i=1}^{m} y_i^T Q_i y_j + R_m, \tag{B3}$$

where

$$R_m = \sum_{i=1}^{m} y_i^T C_i y_j - \sum_{i=1}^{m}\sum_{j=1}^{m} y_i^T B_{ij} y_j = \sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j} (B_{ij})_{rq}\{(y_i)_r - (y_j)_q\}^2, \tag{B4}$$

which is always non-negative because $B_{ij} \geqslant 0$.

*Proof.* The quadratic form $y^T Q y$ is

$$\begin{aligned}
y^T Q y &= \begin{bmatrix} y_1^T & y_2^T & \cdots & y_m^T \end{bmatrix} \begin{bmatrix} Q_1 + C_1 & -B_{12} & \cdots & -B_{1m} \\ -B_{21} & Q_2 + C_2 & \cdots & -B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{m1} & -B_{m2} & \cdots & Q_m + C_m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \\
&= \begin{bmatrix} y_1^T & y_2^T & \cdots & y_m^T \end{bmatrix} \begin{bmatrix} (Q_1 + C_1) y_1 - \sum_{j=2}^{m} B_{1j} y_j \\ (Q_2 + C_2) y_2 - \sum_{j=1; j\neq 2}^{m} B_{2j} y_j \\ \vdots \\ (Q_m + C_m) y_m - \sum_{j=1}^{m-1} B_{mj} y_j \end{bmatrix} \\
&= \sum_{i=1}^{m} y_i^T (Q_i + C_i) y_i - \sum_{i=1}^{m} \sum_{j=1; j\neq i}^{m} y_i^T B_{ij} y_j .
\end{aligned}$$

With the convention that $B_{jj} = O$, the quadratic form reads

$$y^T Q y = \sum_{i=1}^{m} y_i^T Q_i y_j + \sum_{i=1}^{m} y_i^T C_i y_j - \sum_{i=1}^{m}\sum_{j=1}^{m} y_i^T B_{ij} y_j .$$

Since $(y_i^T B_{ij} y_j)^T = y_j^T B_{ij}^T y_i = y_j^T B_{ji} y_i$ and a scalar satisfies $(y_i^T B_{ij} y_j)^T = y_i^T B_{ij} y_j$, we have that $y_i^T B_{ij} y_j = y_j^T B_{ji} y_i$ and

$$\sum_{i=1}^{m}\sum_{j=1}^{m} y_i^T B_{ij} y_j = 2\sum_{i=1}^{m}\sum_{j=1}^{i-1} y_i^T B_{ij} y_j = 2\sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j} \left(B_{ij}\right)_{rq} (y_i)_r \left(y_j\right)_q .$$

Moreover,

$$\begin{aligned}
\sum_{i=1}^{m} y_i^T C_i y_i &= \sum_{i=1}^{m} y_i^T \operatorname{diag}\left(\sum_{j=1}^{m} (B_{ij} u_{n_j})_q\right) y_i = \sum_{i=1}^{m}\sum_{j=1}^{m} y_i^T \left(\operatorname{diag}\left(\sum_{q=1}^{n_j} \left(B_{ij}\right)_{rq}\right)_r\right) y_i = \sum_{i=1}^{m}\sum_{j=1}^{m}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j} \left(B_{ij}\right)_{rq} (y_i)_r^2 \\
&= \sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j} \left(B_{ij}\right)_{rq} (y_i)_r^2 + \sum_{i=1}^{m}\sum_{j=i}^{m}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j} \left(B_{ij}\right)_{rq} (y_i)_r^2
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{m}\sum_{j=i}^{m}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j} (B_{ij})_{rq} (y_i)_r^2 &= \sum_{j=1}^{m}\sum_{i=1}^{j}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j} (B_{ij})_{rq} (y_i)_r^2 = \sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{r=1}^{n_j}\sum_{q=1}^{n_i} (B_{ji})_{rq} (y_j)_r^2 \\
&= \sum_{i=1}^{m}\sum_{j=1}^{i}\sum_{q=1}^{n_i}\sum_{r=1}^{n_j} (B_{ij})_{qr} (y_j)_r^2 = \sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j} (B_{ij})_{rq} (y_j)_q^2 .
\end{aligned}$$

Combining all,

$$\sum_{i=1}^{m} y_i^T C_i y_j - \sum_{i=1}^{m}\sum_{j=1}^{m} y_i^T B_{ij} y_j = \sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j}(B_{ij})_{rq}\{(y_i)_r^2 + (y_j)_q^2 - 2(y_i)_r(y_j)_q\}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{i-1}\sum_{r=1}^{n_i}\sum_{q=1}^{n_j}(B_{ij})_{rq}\{(y_i)_r - (y_j)_q\}^2,$$

which is (B4). Finally, the quadratic form is (B3). ■

If $y = x_N = \frac{u}{\sqrt{N}}$ (and thus $y_i = \frac{u_{n_i}}{\sqrt{N}}$) is the eigenvector belonging to the zero eigenvalue of $Q$, then $Qx_N = 0$ expresses that the row sum of any Laplacian is zero. Since all terms in (B3) are non-negative, we conclude each term must be zero. We know that $y_i^T Q_i y_j = 0$ if $y_i = \alpha_i u_{n_i}$ for an arbitrary constant $\alpha_i$. However, $R_m$ in (B4) is only zero, if each term in the sum in (B4) is zero, which requires that each $y_i = \frac{1}{\sqrt{N}} u_{n_i}$, consistent with $x_N = \frac{u}{\sqrt{N}}$.

**Nontrivial eigenvector and eigenvalue solution**

The goal in this section is to generalize Theorem 3 to an $m$-layer interconnected graph. If each interconnection matrix $B_{ij}$ is regular (i.e., with constant row sum and constant column sum), then the $m \times m$ quotient matrix $Q^\pi$ follows from $Q$ in (B1), in which each block matrix is replaced by a real number equal to the row sum of that block matrix. All eigenvalues of $Q^\pi$ (and corresponding eigenvectors, after a transformation by the community matrix $S$ as shown in Sec. III A) are also eigenmodes of $Q$.

In particular, if $y_i = \alpha_i u_{n_i}$ for an arbitrary constant $\alpha_i$ as suggested by Theorem 3, then $y_i^T Q_i y_j = 0$, but $R_m$ is then positive, equal to $y^T Q y > 0$, and

$$R_m = \sum_{i=1}^{m}\sum_{j=1}^{i-1}\{\alpha_i - \alpha_j\}^2 \sum_{r=1}^{n_i}\sum_{q=1}^{n_j}(B_{ij})_{rq}$$
$$= \sum_{i=1}^{m}\sum_{j=1}^{i}\{\alpha_i - \alpha_j\}^2 u_{n_i}^T B_{ij} u_{n_j}.$$

The normalization (B2) of the $y$ vector then indicates that

$$\sum_{j=1}^{m} \alpha_i^2 n_i = 1, \tag{B5}$$

which represents a ellipsoid in the $m$-dimensional space around the origin. If $y$ is an eigenvector, then the orthogonality of eigenvectors requires that $y^T u = 0$,

$$\sum_{j=1}^{m} \alpha_i n_i = 0 \tag{B6}$$

represent a plane through the origin and orthogonal to the vector $(n_1, n_2, \ldots, n_m)$ containing the sizes (number of nodes) in the subgraphs of $G$. The intersection of the ellipsoid (B5) and the plane (B6) determines the possible vectors $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.

Before proceeding, we must verify whether $y = (\alpha_1 u_{n_1}^T, \alpha_2 u_{n_2}^T, \ldots, \alpha_m u_{n_m}^T)$ with not all $\alpha_i$ constant or zero can be an eigenvector of $Q$ that satisfies $Qy = \mu y$ for some positive $\mu$. We execute $Qy$,

$$\begin{bmatrix} Q_1 + C_1 & -B_{12} & \cdots & -B_{1m} \\ -B_{21} & Q_2 + C_2 & \cdots & -B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -B_{m1} & -B_{m2} & \cdots & Q_m + C_m \end{bmatrix} \begin{bmatrix} \alpha_1 u_{n_1} \\ \alpha_2 u_{n_2} \\ \vdots \\ \alpha_m u_{n_m} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha_1 C_1 u_{n_1} - \sum_{j=2}^{m} B_{1j}\alpha_j u_{n_j} \\ \alpha_2 C_2 u_{n_2} - \sum_{j=1;j\neq 2}^{m} B_{2j}\alpha_j u_{n_j} \\ \vdots \\ \alpha_m C_m u_{n_m} - \sum_{j=1}^{m-1} B_{mj}\alpha_j u_{n_j} \end{bmatrix}.$$

With

$$C_k u_{n_k} = \mathrm{diag}\left(\sum_{j=1}^{m}(B_{kj}u_{n_j})_q\right) u_{n_k} = \sum_{j=1}^{m} \mathrm{diag}((B_{kj}u_{n_j})_q) u_{n_k}$$
$$= \sum_{j=1}^{m} B_{kj} u_{n_j}$$

we find that block row $k$ equals

$$\alpha_k C_k u_{n_k} - \sum_{j=1}^{m} B_{kj}\alpha_j u_{n_j} = \sum_{j=1}^{m}(\alpha_k - \alpha_j) B_{kj} u_{n_j}.$$

Let us assume that $\alpha_k \neq 0$, because there must be at least of the $\alpha_j$ be different from zero (and not all $\alpha_j = c$). In order for this $k$th block vector to be the $k$th block eigenvector of $Q$, there must hold, for some nonzero $\xi$, that[8]

$$\sum_{j=1}^{m}(\alpha_k - \alpha_j) B_{kj} u_{n_j} = \xi \alpha_k u_{n_k}.$$

This requirement is, in general, difficult to verify.

[1] P. Van Mieghem, *Performance Analysis of Complex Networks and Systems* (Cambridge University Press, Cambridge, 2014).

[2] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Catastrophic cascade of failures in interdependent networks, Nat. Lett. **464**, 1025 (2010).

[3] F. D. Sahneh, C. Scoglio, and P. Van Mieghem, Coupling threshold for structural transition in interconnected networks, Phys. Rev. E **92**, 040801 (2015).

[4] M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, and A. Arenas, Mathematical Formulation of Multilayer Networks, Phys. Rev. X **3**, 041022 (2013).

[5] C. J. Stam, E. C. W. van Straaten, E. Van Dellen, P. Tewarie, G. Gong, A. Hillebrand, J. Meier, and P. Van Mieghem, The relation between structural and functional connectivity patterns in complex brain networks, Int. J. Psychophysiol. (to be published).

[6] P. Tewarie, A. Hillebrand, B. W. van Dijk, C. J. Stam, G. C. O'Neill, P. Van Mieghem, J. Meier, M. W. Woolrich, P. G. Morris, and M. J. Brookes, Integrating cross-frequency and within band functional networks in resting-state meg: a multi-layer network approach, NeuroImage (to be published).

[7] P. Van Mieghem, F. D. Sahneh, and C. Scoglio, Exact Markovian SIR and SIS epidemics on networks and an upper bound for the epidemic threshold, *Proceedings of the 53rd IEEE Conference on Decision and Control (CDC2014), Los Angeles* (IEEE, Piscataway, 2014).

[8] P. Van Mieghem, Approximate formula and bounds for the time-varying SIS prevalence in networks (unpublished).

[9] F. Radicchi, Driving interconnected networks to supercriticality, Phys. Rev. X **4**, 021014 (2014).

[10] L. M. Shekhtman, S. Shai, and S. Havlin, Resilience of networks formed of interdependent modular networks, New J. Phys. **17**, 123007 (2015).

[11] H. Wang, Q. Li, G. D'Agostino, S. Havlin, H. E. Stanley, and P. Van Mieghem, Effect of the interconnected network structure on the epidemic threshold, Phys. Rev. E **88**, 022801 (2013).

[12] J. Martin-Hernandez, H. Wang, P. Van Mieghem, and G. D'Agostino, Algebraic connectivity of interdependent networks, Physica A **404**, 92 (2014).

[13] P. Van Mieghem, *Graph Spectra for Complex Networks* (Cambridge University Press, Cambridge, 2011).

[14] P. Van Mieghem and J. Omic, Inhomogeneous virus spread in networks, Delft University of Technology, Report 2008081; arXiv:1306.2588.

[15] P. Van Mieghem, Graph eigenvectors, fundamental weights and centrality metrics for nodes in networks, Delft University of Technology, Report 20150808; arXiv:1401.4580.

[16] F. Radicchi and A. Arenas, Abrupt transition in structural formation of interconnected networks, Nat. Phys. **9**, 717 (2013).

[17] H. Shakeri, N. Albin, F. D. Sahneh, P. Poggi-Corradini, and C. Scoglio, Maximizing algebraic connectivity in interconnected networks, Phys. Rev. E **93**, 030301(R) (2016).

[18] M. Fiedler, Algebraic connectivity of graphs, Czech. Math. J. **23**, 298 (1973).

[19] X. Wang and P. Van Mieghem, Orthogonal eigenvector matrix of the Laplacian, *Fourth International IEEE Workshop on Complex Networks and their Applications, Bangkok, Thailand* (IEEE, Piscataway, 2015).