Linear Algebra and its Applications 437 (2012) 319–323

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Bounds for the spectral radius of a graph when nodes are removed

Cong Li *, Huijuan Wang, Piet Van Mieghem

Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands

ARTICLE INFO

Article history:
Received 29 November 2011
Accepted 26 February 2012
Available online 18 March 2012

Submitted by P. Šemrl

Keywords:
Spectral radius
Eigenvector
Graph
Link
Node

ABSTRACT

We present a new type of lower bound for the spectral radius of a graph in which m nodes are removed. As a corollary, Cioabă's theorem [4], which states that the maximum normalized principal eigenvector component in any graph never exceeds $\frac{1}{\sqrt{2}}$ (with equality for the star), appears as a special case of our more general result.

Introduction

We consider a graph $G = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is the set of nodes and $\mathcal{L}$ is the set of links. The number of nodes is denoted by $N = |\mathcal{N}|$ and the number of links is represented by $L = |\mathcal{L}|$. The graph G can be represented by the $N \times N$ adjacency matrix A, consisting of elements a_{ij} that are either one or zero depending on whether there is a link between node i and j. The eigenvalues of the adjacency matrix A are ordered as $\lambda_N \leqslant \lambda_{N-1} \leqslant \cdots \leqslant \lambda_1$, where λ_1 is the spectral radius and the corresponding eigenvector x_1, normalized such that $x_1^T x_1 = 1$, is called the principal eigenvector. Let $\mathcal{L}_m$ (or $\mathcal{N}_m$) denote the set of the m links (or nodes) that are removed from G, and $G_m(\mathcal{L}) = G \backslash \mathcal{L}_m$ (or $G_m(\mathcal{N}) = G \backslash \mathcal{N}_m$) is the resulting graph after the removal of m links (or nodes) from G. We denote the adjacency matrix of $G_m(\mathcal{L})$ (or $G_m(\mathcal{N})$) by $A_m(\mathcal{L})$ (or $A_m(\mathcal{N})$), which is still a symmetric matrix. Similarly, let w_1 be the normalized eigenvector (as in [9]) of $A_m(\mathcal{L})$ (or $A_m(\mathcal{N})$) corresponding to $\lambda_1(A_m(\mathcal{L}))$ (or $\lambda_1(A_m(\mathcal{N}))$) in the graph $G_m(\mathcal{L})$ (or $G_m(\mathcal{N})$) (such that $w_1^T w_1 = 1$). By the Perron–Frobenius theorem [8], all components of x_1 and w_1 are non-negative (positive if the corresponding graph is connected).

* Corresponding author.
E-mail addresses: Cong.Li@tudelft.nl (C. Li), H.Wang@tudelft.nl (H. Wang), P.F.A.VanMieghemv@tudelft.nl. (P. Van Mieghem).

0024-3795/$ - see front matter
http://dx.doi.org/10.1016/j.laa.2012.02.023

Many inequalities for the spectral radius have been published (see e.g. [7,8]). The search to improve the bounds for the spectral radius will continue due to the intimate relation with dynamic processes such as epidemics and synchronization in networks as explained in [9]. Our main result here is:

Theorem 1. *For any graph G and corresponding graph $G_m(\mathcal{N}) = G\backslash\mathcal{N}_m$, obtained from G by removing the set $\mathcal{N}_m$ of m nodes, it holds that*

$$\left(1 - 2\sum_{n\in\mathcal{N}_m}(x_1)_n^2\right)\lambda_1(A) + \sum_{j\in\mathcal{N}_m}\sum_{i\in\mathcal{N}_m} a_{ij}(x_1)_i(x_1)_j \leqslant \lambda_1\left(A_m(\mathcal{N})\right) \leqslant \lambda_1\left(A\right) \tag{1}$$

where x_1 is the eigenvector of A corresponding to the largest eigenvalue $\lambda_1\ (A)$. In particular, if $m = 1$, then

$$\left(1 - 2\,(x_1)_n^2\right)\lambda_1(A) \leqslant \lambda_1\left(A_1(\mathcal{N})\right) \leqslant \lambda_1\left(A\right) \tag{2}$$

Proof. After removing a node n from graph G, we obtain $A_1(\mathcal{N})$, which is a $(N-1)\times(N-1)$ matrix,

$$A_1(\mathcal{N}) = \begin{bmatrix} a_{11} & \cdots & a_{1(n-1)} & a_{1(n+1)} & \cdots & a_{1N} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} & a_{(n-1)(n+1)} & \cdots & a_{(n-1)N} \\ a_{(n+1)1} & \cdots & a_{(n+1)(n-1)} & a_{(n+1)(n+1)} & \cdots & a_{(n+1)N} \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{N1} & \cdots & a_{N(n-1)} & a_{N(n+1)} & \cdots & a_{NN} \end{bmatrix}$$

Consider the $N \times N$ matrix,

$$\widetilde{A_1}(\mathcal{N}) = \begin{bmatrix} a_{11} & \cdots & a_{1(n-1)} & 0 & a_{1(n+1)} & \cdots & a_{1N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} & 0 & a_{(n-1)(n+1)} & \cdots & a_{(n-1)N} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a_{(n+1)1} & \cdots & a_{(n+1)(n-1)} & 0 & a_{(n+1)(n+1)} & \cdots & a_{(n+1)N} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{N1} & \cdots & a_{N(n-1)} & 0 & a_{N(n+1)} & \cdots & a_{NN} \end{bmatrix}$$

which has the same largest eigenvalue as $A_1(\mathcal{N})$. In fact, all eigenvalues of $A_1(\mathcal{N})$ are the same as in $\widetilde{A_1}(\mathcal{N})$, that possesses an additional zero eigenvalue. In the following deduction, we likewise consider $\widetilde{A_1}(\mathcal{N})$ instead of $A_1(\mathcal{N})$ in order to have the dimension equal to $N \times N$. The principal eigenvector w_1 corresponding to $\lambda_1(A_m(\mathcal{N}))$ is also extended to a vector with N components, where the components corresponding to the removed nodes are all zeros.

The Rayleigh principle states that $x^T Ax \leqslant \lambda_1(A)$ for any normalized vector x with $x^T x = 1$ and equality is only attained when $x = x_1$. Since x_1 is an eigenvector of A, but not necessarily an eigenvector of $\widetilde{A_1}(\mathcal{N})$ belonging to $\lambda_1(\widetilde{A_1}(\mathcal{N}))$, we have that $\lambda_1(\widetilde{A_1}(\mathcal{N})) \geqslant x_1^T(\widetilde{A_1}(\mathcal{N}))x_1$, where

$$x_1^T(\widetilde{A_1}(\mathcal{N}))x_1 = x_1^T A x_1 - x_1^T(A - \widetilde{A_1}(\mathcal{N}))x_1 = \lambda_1(A) - x_1^T(A - \widetilde{A_1}(\mathcal{N}))x_1 \tag{3}$$

It remains to compute $x_1^T(A - \widetilde{A_1}(\mathcal{N}))x_1$. We can write

$$A - \widetilde{A_1}(\mathcal{N}) = a_n \cdot e_n^T + e_n \cdot a_n^T$$

where a_n is the column vector $(a_{n1}, a_{n2}, \ldots, a_{nN})^T$ and e_n is the nth basis column vector $(0, 0, \ldots, 1, \ldots, 0)^T$, where only the nth component is 1. Hence,

$$\begin{aligned} x_1^T(A - \widetilde{A_1}(\mathcal{N}))x_1 &= x_1^T(a_n \cdot e_n^T + e_n \cdot a_n^T)x_1 \\ &= x_1^T a_n e_n^T x_1 + x_1^T e_n a_n^T x_1 = 2(x_1)_n \sum_{i=1}^{N} (x_1)_i a_{in} \end{aligned}$$

The eigenvalue equation written for the component n yields

$$\sum_{i=1}^{N} (x_1)_i a_{in} = \lambda_1(A)(x_1)_n$$

so that we arrive at

$$x_1^T(A - \widetilde{A_1}(\mathcal{N}))x_1 = 2(x_1)_n^2 \lambda_1(A) \tag{4}$$

Introduced in (3) yields the lower bound in (2).

We repeat the analysis from the point of view of $\widetilde{A_1}(\mathcal{N})$. Since w_1 is an eigenvector of $\widetilde{A_1}(\mathcal{N})$, but not necessarily an eigenvector of A belonging to $\lambda_1(A)$, we have $\lambda_1(A) \geqslant w_1^T A w_1$. Similarly as above,

$$\begin{aligned} \lambda_1(A) &\geqslant w_1^T \widetilde{A_1}(\mathcal{N}) w_1 + w_1^T \left(A - \widetilde{A_1}(\mathcal{N})\right) w_1 \\ &= \lambda_1(\widetilde{A_1}(\mathcal{N})) + w_1^T \left(A - \widetilde{A_1}(\mathcal{N})\right) w_1 \\ &= \lambda_1(\widetilde{A_1}(\mathcal{N})) + 2\lambda_1(\widetilde{A_1}(\mathcal{N}))(w_1)_n^2 \end{aligned} \tag{5}$$

from which, with $\sum_{i=1}^{N}(w_1)_i a_{in} = \lambda_1(\widetilde{A_1}(\mathcal{N}))(w_1)_n$ and $a_n = 0$ in $\widetilde{A_1}(\mathcal{N})$ so that $(w_1)_n = 0$, the upper bound in (2) follows.

Next, we extend inequality (3) in case m nodes are removed,

$$x_1^T(A - A_m(\mathcal{N}))x_1 = x_1^T \left(\sum_{n \in \mathcal{N}_m} a_n \cdot e_n^T + \sum_{n \in \mathcal{N}_m} e_n \cdot a_n^T - \sum_{j \in \mathcal{N}_m} \sum_{i \in \mathcal{N}_m} a_{ij} e_i e_j^T \right) x_1$$

and obtain

$$\begin{aligned} \lambda_1(A_m(\mathcal{N})) &\geqslant \lambda_1(A) - x_1^T(A - A_m(\mathcal{N}))x_1 \\ &= \lambda_1(A) - 2\lambda_1(A) \sum_{n \in \mathcal{N}_m} (x_1)_n^2 + \sum_{j \in \mathcal{N}_m} \sum_{i \in \mathcal{N}_m} a_{ij}(x_1)_i(x_1)_j \end{aligned} \tag{6}$$

Similarly, when repeating the analysis from the point of view of $A_m(\mathcal{N})$ rather than from A, we can also extend inequality (5) in case m nodes are removed. With $\lambda_1(A) \geqslant w_1^T(A)w_1$, we achieve

$$\begin{aligned} \lambda_1(A) &\geqslant \lambda_1(A_m(\mathcal{N})) - w_1^T(A_m(\mathcal{N}) - A)w_1 \\ &= \lambda_1(A_m(\mathcal{N})) + 2\lambda_1(A_m(\mathcal{N})) \sum_{n \in \mathcal{N}_m} (w_1)_n^2 - \sum_{j \in \mathcal{N}_m} \sum_{i \in \mathcal{N}_m} a_{ij}(w_1)_i(w_1)_j \end{aligned}$$

with $(w_1)_i = 0$, if $i \in \mathcal{N}_m$,

$$\lambda_1(A) \geqslant \lambda_1(A_m(\mathcal{N})) \tag{7}$$

From the inequality (6) and (7), we arrive at the bounds (1) of $\lambda_1\,(A_m(\mathcal{N}))$. $\square$

The addition of a node to a graph G_N was discussed in [8, p. 60, art. 60]. In particular, when G_{N+1} is the cone of a regular graph G_N, the spectral radius $\lambda_1(A_{N+1})$ of G_{N+1} equals $\frac{\lambda_1(A_N)}{2}\left(1+\sqrt{1+4\frac{d_n}{\lambda_1(A_N)^2}}\right)$, where $\lambda_1(A_N)$ is the spectral radius of G_N and $d_n = N$ is the degree of the added cone node. Hence, the increase of the spectral radius is related to the degree d_n. Lemma 1 shows that the decrease of the spectral radius by removing a node n is related to $(x_1)_n$ and complements a lemma on link removals, proved in [9].

Lemma 1. *For any graph G and $G_m(\mathcal{L}) = G\backslash\mathcal{L}_m$, it holds that*

$$2\sum_{l\in\mathcal{L}_m}(w_1)_{l^+}(w_1)_{l^-} \leqslant \lambda_1(A) - \lambda_1(A_m(\mathcal{L})) \leqslant 2\sum_{l\in\mathcal{L}_m}(x_1)_{l^+}(x_1)_{l^-} \tag{8}$$

where x_1 and w_1 are the eigenvectors of A and A_m corresponding to the largest eigenvalues $\lambda_1(A)$ and $\lambda_1(A_m)$, respectively, and where a link l joins the nodes l^+ and l^-.

Lemma 1 relates the decrease of λ_1 by m link removals to the product $(x_1)_i(x_1)_j$. Moreover, the lower bound in (1) of the spectral radius by removing m nodes contains the term

$$\sum_{j\in\mathcal{N}_m}\sum_{i\in\mathcal{N}_m}a_{ij}(x_1)_i(x_1)_j$$

illustrating that, if there are links between removed nodes (i.e. $l^+ = i$ and $l^- = j$), the decrease of the spectral radius also depends on the product $(x_1)_i(x_1)_j$ over links corresponding to the connected nodes.

In addition, the upper bound in (1) of $\lambda_1(A_m(\mathcal{N}))$ states that the spectral radius λ_1 of a graph G is always larger than or equal to the largest eigenvalue of any subgraph G_s of G,

$$\lambda_1 \geqslant \max_{\text{all } G_s\subset G}(\lambda_1(A_{G_s}))$$

which is another proof for Theorem 42 in [8, pp. 246–247].

Goh et al. [5] observed by simulations in Bárabasi–Albert graphs that the upper bound of $(x_1)^2_{max}$ is $\frac{1}{2}$, where $(x_1)_{max}$ is the largest component of the principal eigenvector. Corollary 1 provides a rigorous proof of this observation.

Corollary 1. *In any graph, any eigenvector component of the principal eigenvector obeys*

$$(x_1)_n \leqslant \frac{\sqrt{2}}{2} \tag{9}$$

Moreover,

$$\sum_{n\in\mathcal{N}_m}(x_1)^2_n \leqslant \frac{1}{2}\left\{1+\frac{1}{\lambda_1(A)}\sum_{j\in\mathcal{N}_m}\sum_{i\in\mathcal{N}_m}a_{ij}(x_1)_i(x_1)_j\right\} \tag{10}$$

Proof. Since all components of x_1 and $\widetilde{A_1}(\mathcal{N})$ are non-negative by the Perron–Frobenius Theorem, we have that $x_1^T(\widetilde{A_1}(\mathcal{N}))x_1 \geqslant 0$. Combining (3), (4) and $\lambda_1(A) > 0$, we obtain $\left(1-2(x_1)^2_n\right) \geqslant 0$, from which (9) follows. By the same argument $x_1^T(\widetilde{A_m}(\mathcal{N}))x_1 \geqslant 0$ and

$$\left(1-2\sum_{n\in\mathcal{N}_m}(x_1)^2_n\right)\lambda_1(A)+\sum_{j\in\mathcal{N}_m}\sum_{i\in\mathcal{N}_m}a_{ij}(x_1)_i(x_1)_j \geqslant 0$$

proving (10). □

Alternatively, the inequality in the proof also yields

$$\lambda_1(A) \geqslant \frac{\sum_{j\in\mathcal{N}_m}\sum_{i\in\mathcal{N}_m} a_{ij}(x_1)_i(x_1)_j}{2\sum_{n\in\mathcal{N}_m}(x_1)_n^2 - 1} = \frac{\sum_{l\in\mathcal{L}_m^*}(x_1)_{l^+}(x_1)_{l^-}}{2\sum_{n\in\mathcal{N}_m}(x_1)_n^2 - 1}$$

where $\mathcal{L}_m^*$ denotes the set of links among the set $\mathcal{N}_m$ of nodes removed from G. The sharpest bound is likely reached when $2\sum_{n\in\mathcal{N}_m}(x_1)_n^2 \gtrapprox 1$.

We remark that equality in (9) is reached for the star, when the node n is the central or hub node. Since scale-free graphs consists of few very high degree nodes, their influence on the eigenvector is close to a star, which explains the observations of Goh et al. [5]. When $\mathcal{N}_m = \mathcal{N}$ or $m = N$, then equality in (10) is obtained. When $\mathcal{N}_m$ is an independent set (i.e. there are no links between the nodes of $\mathcal{N}_m$ such that $a_{ij} = 0$ for any $i, j \in \mathcal{N}_m$), the non-negative double sum in (10) disappears and we find that

$$\sum_{n\in\mathcal{N}_m}(x_1)_n^2 \leqslant \frac{1}{2}$$

This special case of (10) has been proved earlier by Cioabă [4]. Cioabă and Gregory [2] also proved other generalizations of inequality (9) such as $(x_1)_n \leqslant \frac{1}{\sqrt{1+\lambda_1^2/d_n}}$, where d_n is the degree of node n, responding to $(x_1)_n$. Since $\lambda_1 \geqslant \sqrt{\Delta} \geqslant \sqrt{d_n}$ (see [8, pp. 55, art. 54]), where Δ is the maximum degree, the inequality (9) follows. Also, Stevanovic bounds [7] relating λ_1 and Δ were improved in [1,3,6,10].

Finally, the lower bound in (2) underlines the interpretation of a principal eigenvector component as an importance or centrality measure. For, the more important the node n is, the higher the value of $(x_1)_n$, and the larger the possible decrease in spectral radius when this node n is removed.

Acknowledgement

We are grateful to D. Stevanović for pointing us to Cioabă's Theorem.

References

[1] S.M. Cioabă, The spectral radius and the maximum degree of irregular graphs, Electron. J. Combin. 14 (2007) 1–10.
[2] S.M. Cioabă, D.A. Gregory, Principal eigenvectors of irregular graphs, Electron. J. Linear Algebra 16 (2007) 366–379.
[3] S.M. Cioabă, D.A. Gregory, V. Nikiforov, Extreme eigenvalues of nonregular graphs, J. Combin. Theory Ser. B 97 (3) (2007) 483–486.
[4] S.M. Cioabă, A necessary and sufficient eigenvector condition for a connected graph to be bipartite, Electron. J. Linear Algebra (ELA) 20 (2010) 351–353.
[5] K.-L. Goh, B. Kahng, D. Kim, Spectra and eigenvectors of scale-free networks, Phys. Rev. E 64 (2001) 051903
[6] B. Liu, J. Shen, X. Wang, On the largest eigenvalue of non-regular graphs, J. Combin. Theory Ser. B 97 (6) (2007) 1010–1018.
[7] D. Stevanović, The largest eigenvalue of nonregular graphs, J. Combin. Theory Ser. B 91 (2004) 143–146.
[8] P. Van Mieghem, Graph Spectra for Complex Networks, Cambridge University Press, Cambridge, UK, 2011.
[9] P. Van Mieghem, D. Stevanović, F.A. Kuipers, C. Li, R. van de Bovenkamp, D. Liu, H. Wang, Decreasing the spectral radius of a graph by link removals, Phys. Rev. E 84 (1) (2011) 016101
[10] L. Shi, The spectral radius of irregular graphs, Linear Algebra Appl. 431 (1–2) (2009) 189–196.