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Linear Processes on Complex Networks

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This paper studies the dynamics of complex networks with a time-invariant underlying topology, composed of nodes with linear internal dynamics and linear dynamic interactions between them. While graph theory defines the underlying topology of a network, a linear time-invariant state-space model analytically describes the internal dynamics of each node in the network. By combining linear systems theory and graph theory, we provide an explicit analytical solution for the network dynamics in discrete-time, continuous-time and the Laplace domain. The proposed theoretical framework is scalable and allows hierarchical structuring of complex networks with linear processes while preserving the information about network, which makes the approach reversible and applicable to large scale networks.

Keywords: Complex Networks; Large-Scale Networks; Linear Dynamics; Linear Interactions; Network Dynamics.

1. Introduction

Networks are everywhere. Real-world examples of networks are electric power networks, transportation networks, water networks, economic networks, the Internet, the World Wide Web, social networks and biological networks. Dorfler *et al.* [1, 2] applied network concepts on electrical networks. Van Mieghem *et al.* [3] examined resistive networks and provided best spreaders, based on a weighted Laplacian matrix, while Cetinay *et al.* [4] analysed the vulnerability of power networks under targeted attacks. Guimera *et al.* [5] found that the world-wide air transportation network is a small-world network, while Dunne *et al.* [6] discovered that food-web networks are generally not small-world networks. Newman *et al.* [7] used the theory of random graphs with arbitrary degree distribution to model the behavior of a collaboration network of scientists. Topology of the Internet and the World Wide Web was discovered by Faloutsos *et al.* [8]. In the past two decades, the network topology has been deeply studied, for which we refer to the books by Newman [9]; Boccaletti *et al.* [10] and by Van Mieghem [11].

Each network is defined by its underlying topology and the dynamics that take place on the network. The interplay between the network topology and dynamics has been an active field of scientific research in the past two decades [12]. However, Newman [9] observed that the progress in analyzing the structural properties of the network has been faster than the one related to the dynamics taking place on the network. Barzel, Harush *et al.* [13–15] showed that, while many real networks tend to have similar (universal) structural properties, there exist classes of dynamical processes that exhibit fundamentally different flow patterns. The network dynamics depend on both the network topology and the type of dynamic interactions between the nodes.

During last two decades, dynamical processes on complex networks such as phase transitions [16], percolation [17], synchronization [18], diffusion [19], epidemic spreading [20–23], collective behavior [7] and traffic [24] have been intensively researched [12]. The dynamics of the real-world networks are non-linear and their underlying topology is time-varying [25]. However, complex networks with linear dynamics have been intensively researched recently [26, 27], which can be motivated in several ways. Firstly, non-linear dynamics on the networks can be approximated [28] or bounded [20] by the linear dynamics, in most cases. Secondly, the notion of controlling complex networks has become an important research question [27, 29]. In system theory, non-linear system control is a difficult problem, which has been developed on previously well-established linear system control theory [30]. An analogous order of research development is noticeable in network control theory. Several names for the complex networks with linear dynamics have been used in literature, such as networks of agents (dynamical systems) [31], networked multi-input-multi-output (MIMO) systems [32] and complex networked dynamical systems [26]. Mentioned approaches define models of the network with linear processes from the system/control theory point of view.

Here, a general framework for a complex network with linear processes is proposed, where nodes perform heterogeneous, higher-order linear dynamics, with multi-dimensional input and output vectors. The framework is based upon two assumptions: (1) The internal dynamics of the nodes, as well their interactions, are linear and (2) The underlying topology of the network is time-invariant. The framework allows each dynamic interaction between the nodes to be defined locally and independently and results in the most general description of a network of linear processes available in the literature, to our best knowledge. We provide the analytic solution (both in discrete-time, continuous-time and the Laplace domain) for the network dynamics as a whole, in terms of the internal dynamics of the nodes and the underlying graph that couples these linear processes. Thus, we preserve the network perspective. A major novelty is the hierarchical structuring of linear dynamics, in which the lowest level in the hierarchy describes individual linearly interacting processes. After a certain clustering, subnetting or grouping of linear processes (i.e. nodes on the lowest hierarchical level), these clusters can be aggregated on the next higher level of the hierarchy again as a linear process, though with a different linear dynamics. The key property of such nodal aggregation is that no information by condensation is lost! In other words, the aggregated node precisely shows the same linear dynamics as the lower level group of individual nodes. Thus, the linearity preserves information, but allows to shield the lower level interconnection details and enables very large networks to be condensed into a smaller network of interacting aggregated nodes that preserves exactly the linear dynamics! In fact, a network with linear processes of any size can be iteratively condensed into a set of hierarchical layers, in which each layer still presents a desired, aggregated network structure. An example is traffic flows (steered by a linear process) in a small neighbourhood, condensed into a city, while cities can be condensed to countries etc. Another example are different measurement techniques of a same phenomenon, where each technique has its own granularity. As long as those techniques are linear, finer-detail measurement can be aggregated with coarser ones by choosing the proper hierarchical layer that combines them. Although the spread of Corona has not a linear dynamics (but can be linearized [33]), mobile individual traces can be combined with aggregated flows measured by sensor, telecom base-stations, WIFI hotspot and so on.

The present paper does not directly contribute to the control theory. However, the proposed model preserves the network perspective in developing the governing equations. The generality of the proposed model (i.e. the type of one-node dynamics and the interactions between nodes), the reversible scalability of the hierarchical structuring as well the network perspective based governing equations are novelties of this paper. Another important application of the proposed model for networks with linear processes is identification. Suppose that input and output sequences can be measured at certain places in the network during a long enough time period. Linear system identification then allows to determine the exact governing equations (see [34]). Our general framework for linear networked processes hierarchically groups the subnetworks between measurement nodes and the aggregated linear dynamics of these subnetworks can be identified.

The paper is structured as follows. Section 2 introduces basic terminology and notations, while the network dynamics and hierarchical structuring are analysed in Section 3. The concept of Extended graph is introduced in Section 4, while the analytical solution for the network dynamics in the discrete-time domain is provided in Section 5. Finally, we conclude in Section 6.

2. Basic notations

Complex networks have two general features: a graph and a service or function, specified by dynamic processes [35].

2.1 *Network topology*

The underlying structure (topology) of the network is assumed to be time-invariant and is represented by a graph $G(\mathscr{N},\mathscr{L})$. The graph G is defined by a set $\mathscr{N}$ of $N = |\mathscr{N}|$ nodes, representing N systems[1], and by a set $\mathscr{L}$ of L links, that interconnect the systems. The link existence of the graph G is specified by the $N \times N$ adjacency matrix W, where $w_{ij} = 1$ means that there exists a link between node i and node j, otherwise $w_{ij} = 0$. The graph G is assumed to be directed, which implies that the adjacency matrix W is not symmetric in general, i.e. $W \neq W^T$.

A node i of the graph G can also be connected to external nodes. We distinguish two types of external nodes: *input* and *output* nodes. The input nodes provide links to the nodes of the graph G and have zero in-degree, while the output nodes receive links from the nodes of the graph G and have zero out-degree. In contrast to external nodes, we call the nodes and the links of the graph G *internal nodes* and *internal links*, respectively.

There are r input nodes, defined by the set $\mathscr{M}$. The input nodes connect to the internal nodes via *input links*, specified by the $r \times N$ matrix Φ:

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1N} \\ \phi_{21} & \phi_{22} & \dots & \phi_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{r1} & \phi_{r2} & \dots & \phi_{rN} \end{bmatrix} \tag{2.1}$$

where $\phi_{ij} = 1$ defines the existence of an input link between the i-th input and j-th internal node, otherwise $\phi_{ij} = 0$.

There are q output nodes, defined by the set $\mathscr{P}$. We refer the links connecting the internal and output nodes *output links*. The existence of the output links is defined by the $N \times q$ matrix Ψ:

$$\Psi = \begin{bmatrix} \psi_{11} & \psi_{12} & \dots & \psi_{1q} \\ \psi_{21} & \psi_{22} & \dots & \psi_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{N1} & \psi_{N2} & \dots & \psi_{Nq} \end{bmatrix} \tag{2.2}$$

[1]In this work, the words node and system have been used interchangeably

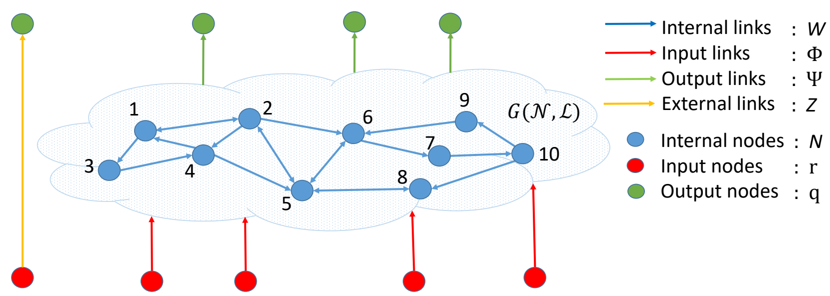

FIG. 1. Different types of nodes and links, in case of a network of 10 nodes

where element ψ_{ij} indicates whether the i-th internal node provides an output link to the j-th output node ($\psi_{ij} = 1$), or not ($\psi_{ij} = 0$).

Finally, each input node can be directly connected to an output node as well. We refer to such a link as *external link* and define their existence with the $r \times q$ matrix Z:

$$Z = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1q} \\ z_{21} & z_{22} & \cdots & z_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rq} \end{bmatrix} \tag{2.3}$$

where element z_{ij} defines whether there is an external link between the input node i and the output node j ($z_{ij} = 1$) or not ($z_{ij} = 0$).

The in-degree of the i-th output node is $(u^T\Psi)_i + (u^T Z)_i$, while the j-th input node has the out-degree $(\Phi u)_j + (Zu)_j$, where u is the all-one vector. All types of nodes and links defined above are presented in Fig. 1 and labelled by a different colour, for a graph G of 10 nodes, with additional 5 input and 4 output nodes.

2.2 *Processes on the network*

Each node in the network is a linear time-invariant (LTI) system, whose dynamics are defined by a discrete-time linear state space (DLSS) model [36]. The dynamics within the i-th node/system obey the DLSS governing equations:

$$\begin{cases} x_i[k+1] & = A_i \cdot x_i[k] + B_i \cdot u_i[k] \\ y_i[k] & = C_i \cdot x_i[k] + D_i \cdot u_i[k] \end{cases} \tag{2.4}$$

where the discrete time is modelled by k. The $n_i \times n_i$ state matrix A_i defines how the $n_i \times 1$ state vector x_i depends on its previous value, while the $n_i \times m_i$ input matrix B_i determines the relation between the state vector x_i and the previous value of the $m_i \times 1$ input vector u_i. The relation between the $p_i \times 1$ output vector y_i and the state vector x_i is defined by the $p_i \times n_i$ output matrix C_i. Finally, direct relation between the output vector y_i and the input vector u_i is defined by the $p_i \times m_i$ feedforward matrix D_i.

The interconnected DLSS dynamics are sketched in Fig. 2, in case of a network with three nodes. We define the $N \times 1$ vector n, containing the number of states for each node/system of the network:

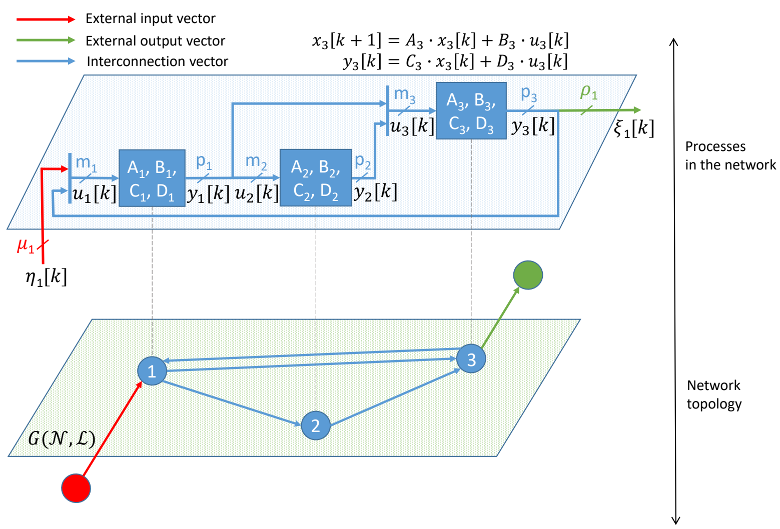

FIG. 2. DLSS dynamics of a simple network with $N = 3$ nodes/systems

$$n = \begin{bmatrix} n_1 & n_2 & \dots & n_i & \dots & n_N \end{bmatrix}^T \tag{2.5}$$

Similarly, we define the $N \times 1$ vector m that contains the dimension of the input vector u_i for each system $(i \in \mathcal{N})$:

$$m = \begin{bmatrix} m_1 & m_2 & \dots & m_i & \dots & m_N \end{bmatrix}^T \tag{2.6}$$

where m_i represents the dimension of the input vector u_i of the node/system i. Analogously, the $N \times 1$ vector p defines the dimension of the output vector y_i for each system in the network $(i \in \mathcal{N})$:

$$p = \begin{bmatrix} p_1 & p_2 & \dots & p_i & \dots & p_N \end{bmatrix}^T \tag{2.7}$$

where p_i represents the dimension of the output vector y_i of the i-th system.

The input vector u_i of the i-th system of the network can be composed of the output vectors from other systems (due to interconnections) and of the external input vectors. In other words, only internal and input links can be connected to an internal node.

The i-th external input vector is denoted by η_i and has dimension $\mu_i \times 1$. We define the $r \times 1$ vector μ that contains the dimension of each external input vector:

$$\mu = \begin{bmatrix} \mu_1 & \mu_2 & \dots & \mu_i & \dots & \mu_r \end{bmatrix}^T \tag{2.8}$$

In addition, we define the $M \times 1$ vector η, by concatenating r external input vectors:

$$\eta = \begin{bmatrix} \eta_1 & \eta_2 & \dots & \eta_i & \dots & \eta_r \end{bmatrix}^T \tag{2.9}$$

where $M = \sum_{j=1}^{r} \mu_j$.

An external output vector can be composed of the output vectors of the systems from the network, as well as of the external input vectors. The i-th external output vector is denoted by ξ_i and has dimension $\rho_i \times 1$. We define the $q \times 1$ vector ρ containing the dimension of each external output vector ξ_i ($i \in \mathscr{P}$):

$$\rho = \begin{bmatrix} \rho_1 & \rho_2 & \dots & \rho_i & \dots & \rho_q \end{bmatrix}^T \tag{2.10}$$

In addition, we define the $P \times 1$ vector ξ, composed by concatenating q external output vectors:

$$\xi = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_i & \dots & \xi_q \end{bmatrix}^T \tag{2.11}$$

where $P = \sum_{j=1}^{q} \rho_j$. We introduce the $N \times 1$ vectors l_ϕ and l_w, as well as the $q \times 1$ vectors l_z and l_ψ as

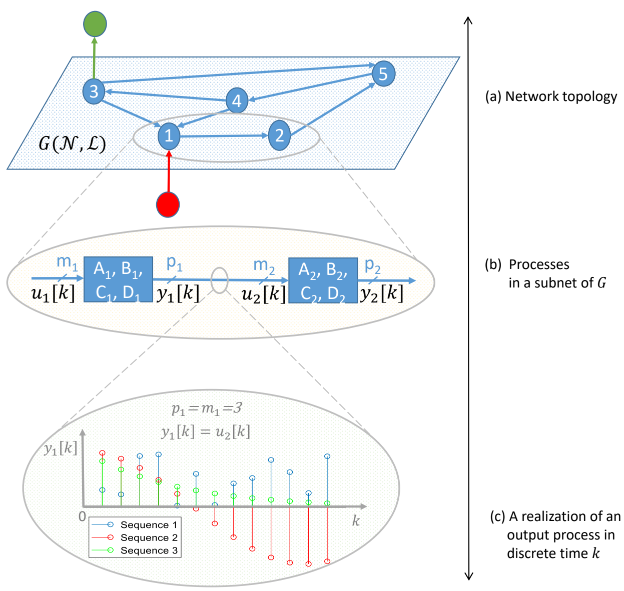

FIG. 3. Network topology, processes and time realization of a process

follows:

$$\begin{aligned} l_\phi &= \Phi^T \cdot u_{r\times 1} \qquad & l_w &= W^T \cdot u_{N\times 1} \\ l_z &= Z^T \cdot u_{r\times 1} \qquad & l_\psi &= \Psi^T \cdot u_{N\times 1} \end{aligned} \tag{2.12}$$

where $(l_w)_i$ defines the number of internal links connected to the internal node i, while the number of input links that internal node i receives is defined by $(l_\psi)_i$. Additionally, the output node i receives $(l_\psi)_i$ links from the internal nodes, as well as $(l_z)_i$ external links. Total number of the internal, input, output and external links is defined as follows:

$$\begin{aligned} L_w &= \left(l_w\right)^T \cdot u_{N\times 1} \qquad & L_\phi &= \left(l_\phi\right)^T \cdot u_{N\times 1} \\ L_\psi &= \left(l_\psi\right)^T \cdot u_{q\times 1} \qquad & L_z &= \left(l_z\right)^T \cdot u_{q\times 1} \end{aligned} \tag{2.13}$$

respectively.

A graph G of 5 nodes, together with the one input and one output node is presented in Fig. 3(a). The processes within the first and second node of G are sketched in Fig. 3(b). Finally, a realization of the output vector of the first node is presented in Fig. 3(c).

The dimension m_i of the input vector u_i of each node in G ($i \in \mathscr{N}$) must obey:

$$m = W^T \cdot p + \Phi^T \cdot \mu \tag{2.14}$$

Analogously, the dimension ρ_i of each external output vector ξ_i ($i \in \mathscr{P}$) must obey:

$$\rho = \Psi^T \cdot p + Z^T \cdot \mu \tag{2.15}$$

Relations (2.14) and (2.15) can be written together in a matrix form:[2]

$$\begin{bmatrix} m_{N\times 1} \\ \rho_{q\times 1} \end{bmatrix} = \begin{bmatrix} W^T_{N\times N} & \Phi^T_{N\times r} \\ \Psi^T_{q\times N} & Z^T_{q\times r} \end{bmatrix} \cdot \begin{bmatrix} p_{N\times 1} \\ \mu_{r\times 1} \end{bmatrix} \tag{2.16}$$

3. Network dynamics

A complex network is composed of N nodes/systems, with internal DLSS dynamics defined by (2.4). We now would like to find the dynamics between the aggregated external output vector ξ defined in (2.11) and the aggregated external input vector η defined in (2.9), by following DLSS governing equations:

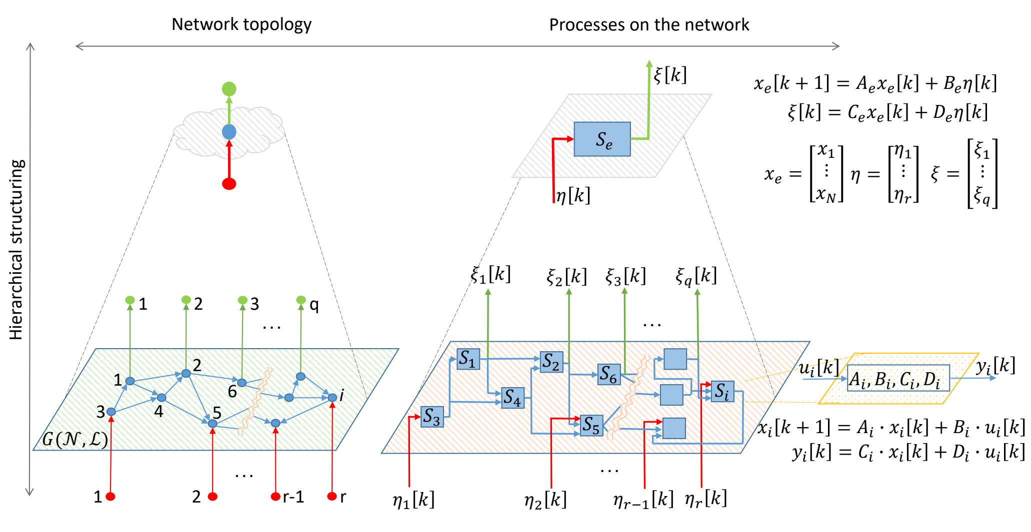

FIG. 4. Underlying topology vs linear processes on the complex network

[2]Determining the rank of the matrix in (2.16) is a problem similar to the problem of determining the rank of the adjacency matrix of a directed graph, see e.g. [37].

$$\begin{cases} x_e[k+1] & = A_e \cdot x_e[k] + B_e \cdot \eta[k] \\ \xi[k] & = C_e \cdot x_e[k] + D_e \cdot \eta[k] \end{cases} \tag{3.1}$$

where the $\sum_{j=1}^{N} n_j \times 1$ vector x_e contains states of each system in the network:

$$x_e[k] = \begin{bmatrix} x_1[k] \\ x_2[k] \\ \vdots \\ x_N[k] \end{bmatrix} \tag{3.2}$$

The matrices A_e, B_e, C_e and D_e will be determined in terms of network topology and the dynamics of individual nodes/systems.

3.1 *Hierarchical Structuring of Complex Networks*

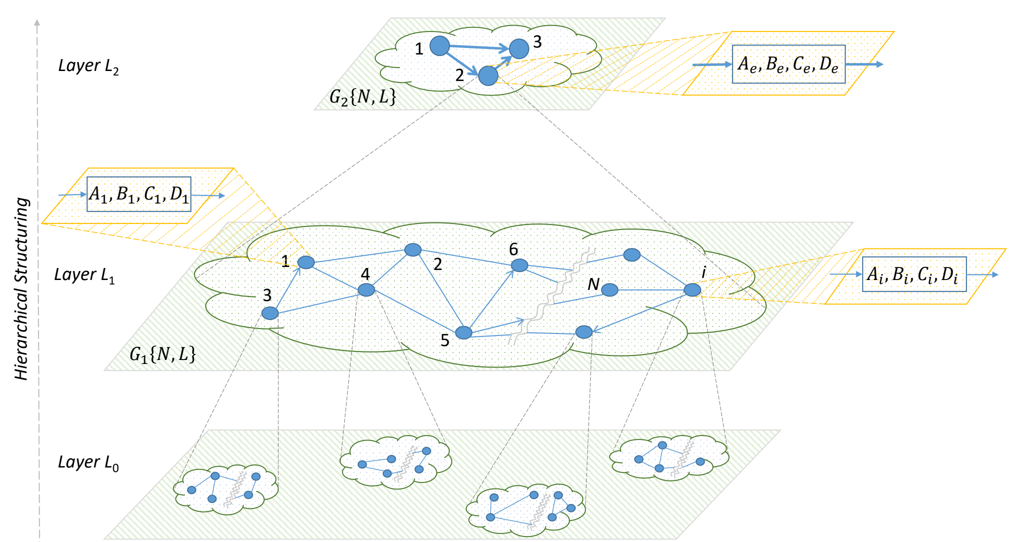

FIG. 5. Hierarchical Structuring of Complex Networks with Linear Processes

The underlying topology of the network, together with the input and output links is sketched in the left lower part of Fig 4, while the processes within each node/system of the network are presented in the right lower part. By determining the DLSS process in (3.1), we determine the network dynamics. Thus, we can abstract the network dynamics with a DLSS process, as provided in the right upper part of Fig 4. This abstraction is analogous to abstracting the network topology by a node, as shown in the left upper part of Fig 4.

An example of hierarchical structuring is provided in Fig 5. We use three layers of abstraction, namely Layer L_0, Layer L_1 and Layer L_2. A network G_1 of N interconnected nodes with internal dynamics is presented in the Layer L_1. The dynamics of the network G_1 are abstracted by the dynamics within the node 2 of the network G_2, in a higher abstraction layer L_2. There are two additional nodes in G_2 and they abstract the dynamics of another two networks from the Layer L_1. An external impact on the network dynamics from the layer L_1 represents an interconnection between the nodes/abstracted networks in G_2.

In the same time, the internal dynamics of a node from G_1 abstract the dynamics of a network from a lower abstraction layer L_0, as presented in Fig 5. An external impact on the dynamics of a network in abstraction layer L_0 represents an interconnection in abstraction layer L_1 and a mode of the dynamics within a node from abstraction layer L_2.

Thus, the proposed theoretical framework allows the hierarchical structuring of complex networks with linear processes. By using the same type of governing equations (DLSS governing equations) to describe both the internal dynamics within a node/system from the network and the network dynamics, we enable hierarchical structuring of complex networks.

4. Extended graph

The underlying topology of the network is defined by a graph G. Beside nodes of the graph G, input

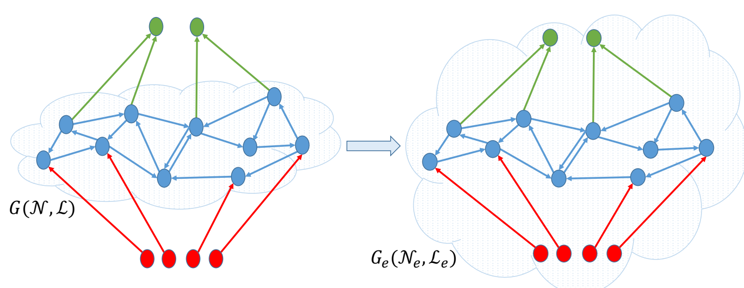

FIG. 6. Concept of the *extended* graph G_e

and output nodes are also defined, as source of input and external links and as destination of output and external links, respectively. Therefore, we introduce the *extended graph* $G_e(\mathscr{N}_e, \mathscr{L}_e)$, that is composed of $N_e = |\mathscr{N}_e|$ nodes:

$$N_e = r + N + q \quad \mathscr{N}_e = \mathscr{M} \cup \mathscr{N} \cup \mathscr{P} \tag{4.1}$$

and of L_e links:

$$L_e = L_\phi + L_w + L_\psi + L_z \tag{4.2}$$

The relation between the graph G and the extended graph G_e is presented in Fig. 6. The input nodes of the extended graph G_e are labelled first, before the internal nodes, while the output nodes are labelled as the last q nodes of G_e. Extended graph G_e from Fig. 6 with labelled nodes is presented in Fig. 7.

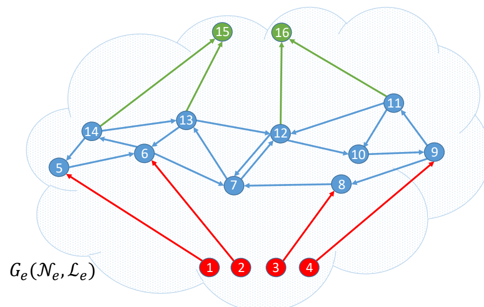

FIG. 7. Labelled nodes of the extended graph G_e from Fig. 6

The adjacency matrix W_e of the extended graph G_e has a block structure:

$$W_e = \begin{bmatrix} O_{r\times r} & \Phi_{r\times N} & Z_{r\times q} \\ O_{N\times r} & W_{N\times N} & \Psi_{N\times q} \\ O_{q\times r} & O_{q\times N} & O_{q\times q} \end{bmatrix} \tag{4.3}$$

Since the input nodes have zero in-degree, the first block column of W_e is composed of zero block matrices. Similarly, since the output nodes have zero out-degree, the third block row contains zero block matrices as well.

The links whose destination is the first **internal** node of G_e are labelled first, in ascending order, relative to the source node. Next, the links connected to the second internal node are labelled. After labelling all the incoming links to the internal nodes, links whose destination is the first **output** node are labelled, in ascending order, relative to the source node. Then the incoming links of the second output node are labelled, in ascending order relative to the source node. The incoming links of the q-th output node are labelled last. The links of the extended graph G_e from Fig. 6 have been labelled by our convention and presented in Fig. 8.

We introduce the $N_e \times L_e$ incidence matrix Λ of extended graph G_e in block form:

$$\Lambda = \begin{bmatrix} (\Lambda_{11})_{r\times(L_w+L_\phi)} & (\Lambda_{12})_{r\times(L_\psi+L_z)} \\ (\Lambda_{21})_{N\times(L_w+L_\phi)} & (\Lambda_{22})_{N\times(L_\psi+L_z)} \\ (\Lambda_{31})_{q\times(L_w+L_\phi)} & (\Lambda_{32})_{q\times(L_\psi+L_z)} \end{bmatrix} \tag{4.4}$$

The first block column of Λ refers to the links whose destination is an internal node. There are $L_w + L_\phi$ such links. The second block column of Λ refers to the links whose destination is an output node. There are $L_\psi + L_z$ such links. The first block row of Λ refers to the r input nodes. Further, the second block row is related to the N internal nodes, while the third block row regards the q output nodes.

We define the $L_e \times N_e$ matrix Γ as follows:

$$\Gamma = \frac{\Lambda^T + |\Lambda^T|}{2} \tag{4.5}$$

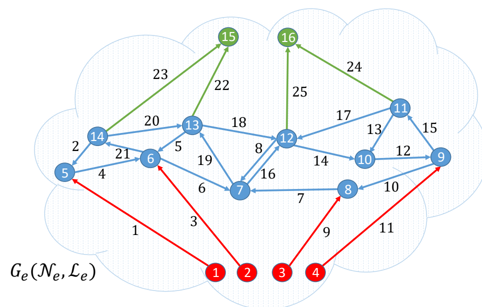

FIG. 8. Labelled links of the extended graph G_e from Fig. 6

where $|\Lambda^T|$ denotes the absolute value of each element of Λ^T. Matrix Γ has a block structure:

$$\Gamma = \begin{bmatrix} (\Gamma_\phi)_{(L_w+L_\phi)\times r} & (\Gamma_w)_{(L_w+L_\phi)\times N} & O_{(L_w+L_\phi)\times q} \\ (\Gamma_z)_{(L_\psi+L_z)\times r} & (\Gamma_\psi)_{(L_\psi+L_z)\times N} & O_{(L_\psi+L_z)\times q} \end{bmatrix} \tag{4.6}$$

where each block element of Γ is of same dimensions as the according block element of transposed incidence matrix Λ^T of G_e. The negative entries of Λ^T define the destination node for each link of G_e and are not contained in Γ. Therefore, the third block column of Γ that is related to output nodes, contains zero block matrices. We observe that matrix Γ is a zero-one matrix. Each row of Γ regards certain link in G_e and contains exact one non-zero component, that refers to the source node of that link.

We further introduce the $(L_w + L_\phi) \times 1$ vectors s_ϕ and s_w, as well as the $(L_\psi + L_z) \times 1$ vectors s_z and s_ψ as follows:

$$\begin{bmatrix} (s_\phi)_{(L_w+L_\phi)\times 1} & (s_w)_{(L_w+L_\phi)\times 1} \\ (s_z)_{(L_\psi+L_z)\times 1} & (s_\psi)_{(L_\psi+L_z)\times 1} \end{bmatrix} = \Gamma \cdot \begin{bmatrix} \mu_{r\times 1} & O_{r\times 1} \\ O_{N\times 1} & p_{N\times 1} \\ O_{q\times 1} & O_{q\times 1} \end{bmatrix} \tag{4.7}$$

where $(s_w)_i$ defines the dimension of the i-th internal link, $(s_\phi)_i$ defines the dimension of the i-th input link, while $(s_\psi)_i$ and $(s_z)_i$ define the dimensions of the i-th output and i-th external link, respectively. The total number of links that are connected to the internal nodes is $L_\phi + L_w$, while $L_\psi + L_z$ is the total number of links that have the output nodes as destination. Total dimensions S_w, S_ϕ, S_ψ and S_z of all internal, input, output and external links, respectively, are defined as follows:

$$\begin{aligned} S_w &= s_w^T \cdot u_{(L_w+L_\phi)\times 1} & S_\phi &= s_\phi^T \cdot u_{(L_w+L_\phi)\times 1} \\ S_\psi &= s_\psi^T \cdot u_{(L_\psi+L_z)\times 1} & S_z &= s_z^T \cdot u_{(L_\psi+L_z)\times 1} \end{aligned} \tag{4.8}$$

Since the input and internal links are connected to internal nodes, while the output and external links have output nodes as a destination, next identities hold:

$$S_w + S_\phi = \textstyle\sum_{i=1}^N m_i \qquad S_\psi + S_z = \sum_{i=1}^q \rho_i \tag{4.9}$$

Additionally, we define the diagonal block matrices containing DLSS matrices of each system of the network, namely $\left(A_d\right)_{\sum_{i=1}^N n_i \times \sum_{i=1}^N n_i}$, $\left(B_d\right)_{\sum_{i=1}^N n_i \times \sum_{i=1}^N m_i}$, $\left(C_d\right)_{\sum_{i=1}^N p_i \times \sum_{i=1}^N n_i}$ and $\left(D_d\right)_{\sum_{i=1}^N p_i \times \sum_{i=1}^N m_i}$:

$$\begin{cases} A_d & = \text{diagonal}\begin{bmatrix} A_1 & A_2 & \dots & A_N \end{bmatrix} \\ B_d & = \text{diagonal}\begin{bmatrix} B_1 & B_2 & \dots & B_N \end{bmatrix} \\ C_d & = \text{diagonal}\begin{bmatrix} C_1 & C_2 & \dots & C_N \end{bmatrix} \\ D_d & = \text{diagonal}\begin{bmatrix} D_1 & D_2 & \dots & D_N \end{bmatrix} \end{cases} \tag{4.10}$$

Matrices A_d, B_d, C_d and D_d enable us to define the dynamics of each system of the network in a compact block diagonal form:

$$\begin{cases} x_e[k+1] & = A_d \cdot x_e[k] + B_d \cdot u_d[k] \\ y_d[k] & = C_d \cdot x_e[k] + D_d \cdot u_d[k] \end{cases} \tag{4.11}$$

where the $\sum_{i=1}^N m_i \times 1$ aggregated input vector u_d and the $\sum_{i=1}^N p_i \times 1$ aggregated output vector y_d are defined as follows:

$$u_d \quad = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \quad y_d = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \tag{4.12}$$

DEFINITION 4.1 The aggregated input vector u_d, aggregated output vector y_d, aggregated external input vector η and the aggregated external output vector ξ are related as follows:

$$\begin{cases} u_d[k] & = F_w \cdot y_d[k] + F_\phi \cdot \eta[k] \\ \xi[k] & = F_\psi \cdot y_d[k] + F_z \cdot \eta[k] \end{cases} \tag{4.13}$$

where the $(S_w + S_\phi) \times \sum_{i=1}^N p_i$ matrix F_w, the $(S_w + S_\phi) \times M$ matrix F_ϕ, the $(S_\psi + S_z) \times \sum_{i=1}^N p_i$ matrix F_ψ and the $(S_\psi + S_z) \times M$ matrix F_z, are composed of $(L_w + L_\phi) \times N$, $(L_w + L_\phi) \times r$, $(L_\psi + L_z) \times N$ and $(L_\psi + L_z) \times r$ block elements, respectively, that are defined as follows:

$$\begin{aligned} \left(F_w\right)_{ij} &= \begin{cases} I_{(s_w+s_\phi)_i} & \text{if } \left(\mathit{\Gamma}_w\right)_{ij} = 1 \\ O_{(s_w+s_\phi)_i \times p_j} & \text{otherwise} \end{cases} \quad \left(F_\phi\right)_{ij} = \begin{cases} I_{(s_w+s_\phi)_i} & \text{if } \left(\mathit{\Gamma}_\phi\right)_{ij} = 1 \\ O_{(s_w+s_\phi)_i \times \mu_j} & \text{otherwise} \end{cases} \\ \left(F_\psi\right)_{ij} &= \begin{cases} I_{(s_\psi+s_z)_i} & \text{if } \left(\mathit{\Gamma}_\psi\right)_{ij} = 1 \\ O_{(s_\psi+s_z)_i \times p_j} & \text{otherwise} \end{cases} \quad \left(F_z\right)_{ij} = \begin{cases} I_{(s_\psi+s_z)_i} & \text{if } \left(\mathit{\Gamma}_z\right)_{ij} = 1 \\ O_{(s_\psi+s_z)_i \times \mu_j} & \text{otherwise} \end{cases} \end{aligned} \tag{4.14}$$

The definition is elaborated in Appendix B. Matrices F_w, F_ϕ, F_ψ and F_z are defined similarly as the Kronecker products. However, each block element of these matrices is of different dimensions, which is not the case in the Kronecker product. Dimensions of the block elements vary because each vector in the network is in general of a different dimension, which is thoroughly explained in Appendix F. Furthermore, in Appendix D, we analyse homogeneous networks with identical dynamic interactions, which is a special case of the network, that allows applying the Kronecker product. Therefore, governing equations for the time dynamics of the entire network are based on the Kronecker product.

5. Main results

THEOREM 5.1 The matrices A_e, B_e, C_e and D_e from the DLSS governing equations in (3.1):

$$\begin{cases} x_e[k+1] & = A_e \cdot x_e[k] + B_e \cdot \eta[k] \\ \xi[k] & = C_e \cdot x_e[k] + D_e \cdot \eta[k] \end{cases}$$

provided the matrix $(I - D_d \cdot F_w)$ is non-singular or $(D_d \cdot F_w)$ has not an eigenvalue 1, are explicitly determined as follows:

$$\begin{cases} A_e = (B_d \cdot F_w) \cdot (I - D_d \cdot F_w)^{-1} \cdot C_d + A_d \\ B_e = (B_d \cdot F_w) \cdot (I - D_d \cdot F_w)^{-1} \cdot (D_d \cdot F_\phi) + B_d \cdot F_\phi \\ C_e = F_\psi \cdot (I - D_d \cdot F_w)^{-1} \cdot C_d \\ D_e = F_\psi \cdot (I - D_d \cdot F_w)^{-1} \cdot (D_d \cdot F_\phi) + F_z \end{cases} \tag{5.1}$$

Proof. Appendix C

COROLLARY 5.1 When there is no direct interaction between the input vector u_i and the output vector y_i of each system in the network (i.e. $D_i = O_{p_i \times m_i}, i \in \mathcal{N}$), the matrices A_e, B_e, C_e and D_e are explicitly determined as follows:

$$\begin{cases} A_e & = B_d \cdot F_w \cdot C_d + A_d \\ B_e & = B_d \cdot F_\phi \\ C_e & = F_\psi \cdot C_d \\ D_e & = F_z \end{cases} \tag{5.2}$$

When the feedforward matrix D_i of each node/system of G is a non zero matrix (i.e. $D_i \neq O_{p_i \times m_i}, i \in \mathcal{N}$), the state vector x_i impacts the state vector x_j (i.e. $\left(A_e\right)_{ji} \neq O_{n_j \times n_i}$) if and only if there is a path from the node i to the node j in G (i.e. iff $\left(\sum_{k=1}^{N} W^k\right)_{ij} > 0$).

On the other side, when there is no direct relation between the input vector u_i and the output vector y_i for each node/system of the network (i.e. $D_i = O_{p_i \times m_i}, i \in \mathcal{N}$), the state vector x_i influences the state vector x_j (i.e. $\left(A_e\right)_{ji} \neq O_{n_j \times n_i}$) if and only if the node/system i and node/system j are direct neighbours (i.e. $w_{ij} = 1$). Thus, the relation (5.2) is significantly simpler than the solution of general case (5.1). The further explanation of the matrices A_e, B_e, C_e and D_e in terms of paths in G_e is provided in Appendix F.

The analysis of the continuous-time process on complex networks is provided Appendix E. The solution for the network dynamics is provided both in time domain and in complex Laplace domain.

5.1 *A numerical example*

We provide a numerical example of a network model with linear processes, on which we apply the results of the paper. Therefore, we provide a network of $N = 5$ nodes/systems, with $r = 2$ input nodes and $q = 2$ output nodes. Further, the $N \times N$ adjacency matrix W, the $r \times N$ matrix Φ, the $N \times q$ matrix Ψ and the $r \times q$ matrix Z are defined as follows:

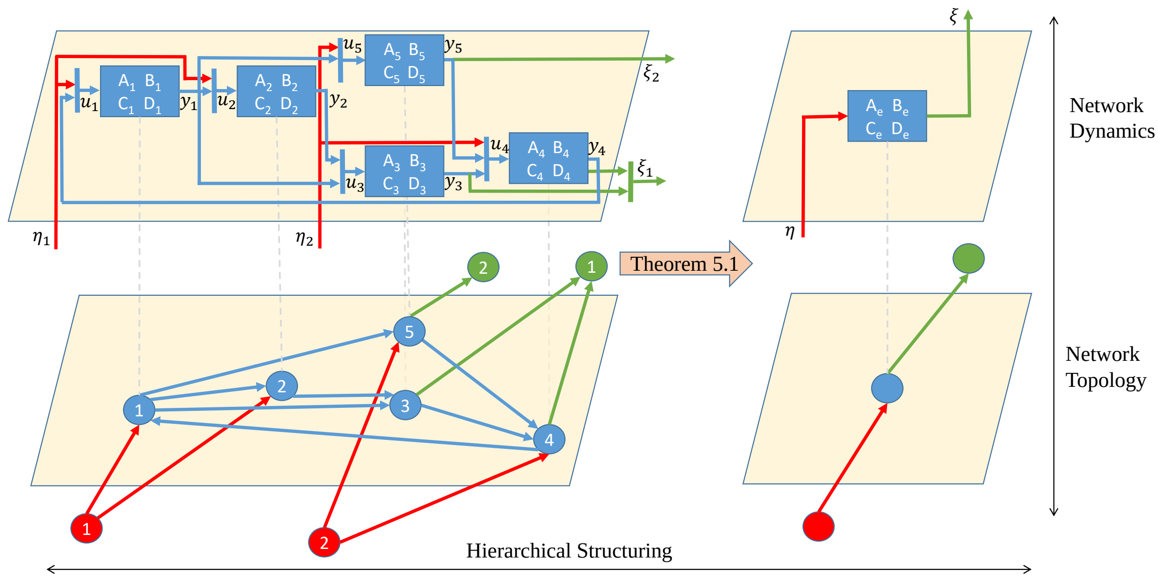

FIG. 9. Network topology and the block diagram of the network dynamics, where $N = 5$, $r = 2$, $q = 2$

$$W = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \Psi = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.3}$$

Further, the $N \times 1$ vector p containing output dimensions of each system, the $N \times 1$ vector n with number of states for each system and the $r \times 1$ vector μ that contains the dimension of external inputs are defined as follows:

$$p = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad n = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \end{bmatrix} \quad \mu = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{5.4}$$

while the $N \times 1$ vector m with dimensions of inputs per each node/system and the $q \times 1$ vector ρ containing dimensions of external outputs are computed using (2.16):

$$m = \begin{bmatrix} 2 & 2 & 2 & 3 & 2 \end{bmatrix} \quad \rho = \begin{bmatrix} 2 & 1 \end{bmatrix} \tag{5.5}$$

Parameters of the DLSS model of each node of the graph are defined below:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} 0.1227 & -0.0733 \\ 0.0733 & 0.1227 \end{bmatrix} & B_1 &= \begin{bmatrix} 0.0232 & 0.1070 \\ 0.1019 & 0.1026 \end{bmatrix} \\
C_1 &= \begin{bmatrix} 0.6547 & 0.1913 \end{bmatrix} & D_1 &= \begin{bmatrix} 0 & 0.5000 \end{bmatrix} \\
A_2 &= \begin{bmatrix} 0.3796 & -0.3920 \\ 0.3920 & 0.3796 \end{bmatrix} & B_2 &= \begin{bmatrix} 0.2371 & 0.1215 \\ 0.4789 & 0.3491 \end{bmatrix} \\
C_2 &= \begin{bmatrix} 0.0089 & 0.1603 \end{bmatrix} & D_2 &= \begin{bmatrix} 0.5000 & 0 \end{bmatrix} \\
A_3 &= \begin{bmatrix} -0.3438 & -0.2597 \\ -0.2597 & -0.7647 \end{bmatrix} & B_3 &= \begin{bmatrix} 0.0597 & 0.3568 \\ 0.4729 & 0.7413 \end{bmatrix} \\
C_3 &= \begin{bmatrix} 0.0664 & 0.2628 \end{bmatrix} & D_3 &= \begin{bmatrix} 0.5000 & 0.3394 \end{bmatrix} \\
A_4 &= \begin{bmatrix} -0.3773 & -0.0779 \\ -0.0779 & -0.9613 \end{bmatrix} & B_4 &= \begin{bmatrix} 0.7038 & 0.6134 & 0.4158 \\ 0.2989 & 0.1345 & 0.5020 \end{bmatrix} \\
C_4 &= \begin{bmatrix} 0.4997 & 0.2145 \end{bmatrix} & D_4 &= \begin{bmatrix} 0 & 0.5000 & 0 \end{bmatrix} \\
A_5 &= \begin{bmatrix} 0.5796 & -0.0619 \\ -0.0619 & 0.7033 \end{bmatrix} & B_5 &= \begin{bmatrix} 0.1072 & 0.6859 \\ 0.4328 & 0.1584 \end{bmatrix} \\
C_5 &= \begin{bmatrix} 0.1089 & 0.0430 \end{bmatrix} & D_5 &= \begin{bmatrix} 0.5000 & 0 \end{bmatrix}
\end{aligned}
\tag{5.6}
$$

Network topology, with input and output nodes, is presented in the lower-left part of Fig. 9, while the network dynamics in from of the interconnected block diagrams are presented in the upper-left part of the Figure. By applying Theorem 5.1 we provide the dynamics of the entire network in the form of a DLSS system, as presented in the upper-right part of the Figure. Finally, the Theorem 5.1 allows representing entire network topology as a node, on a higher hierarchy level.

$$
\begin{aligned}
A_e &= \begin{bmatrix}
0.1402 & -0.0682 & 0.0002 & 0.0029 & 0.0040 & 0.0158 & 0.0601 & 0.0258 & 0 & 0 \\
0.0901 & 0.1276 & 0.0002 & 0.0028 & 0.0038 & 0.0152 & 0.0577 & 0.0248 & 0 & 0 \\
0.0895 & 0.0261 & 0.3797 & -0.3903 & 0.0020 & 0.0080 & 0.0341 & 0.0147 & 0 & 0 \\
0.2571 & 0.0751 & 0.3922 & 0.3843 & 0.0058 & 0.0229 & 0.0981 & 0.0421 & 0 & 0 \\
0.0440 & 0.0128 & 0.0032 & 0.0580 & -0.3428 & -0.2558 & 0.0168 & 0.0072 & 0 & 0 \\
0.3483 & 0.1017 & 0.0069 & 0.1253 & -0.2518 & -0.7336 & 0.1329 & 0.0570 & 0 & 0 \\
0.2259 & 0.0660 & 0.0021 & 0.0375 & 0.0458 & 0.1813 & -0.3006 & -0.0451 & 0.0453 & 0.0179 \\
0.0495 & 0.0145 & 0.0005 & 0.0082 & 0.0100 & 0.0398 & -0.0611 & -0.9541 & 0.0547 & 0.0216 \\
0.5052 & 0.1476 & 0.0005 & 0.0093 & 0.0114 & 0.0451 & 0.1928 & 0.0827 & 0.5796 & -0.0619 \\
0.1167 & 0.0341 & 0.0001 & 0.0022 & 0.0026 & 0.0104 & 0.0445 & 0.0191 & -0.0619 & 0.7033
\end{bmatrix} &
B_e &= \begin{bmatrix}
0.0323 & 0 \\
0.1106 & 0 \\
0.2423 & 0 \\
0.4937 & 0 \\
0.1809 & 0 \\
0.3907 & 0 \\
0.1171 & 0.9117 \\
0.0257 & 0.5499 \\
0.0291 & 0.1072 \\
0.0067 & 0.4328
\end{bmatrix} \\
C_e &= \begin{bmatrix}
0.3683 & 0.1076 & 0.0034 & 0.0612 & 0.0747 & 0.2956 & 0.1406 & 0.0603 & 0 & 0 \\
0.1841 & 0.0538 & 0.0017 & 0.0306 & 0.0374 & 0.1478 & 0.5622 & 0.2413 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1089 & 0.0430
\end{bmatrix} &
D_e &= \begin{bmatrix}
0.1909 & 0 \\
0.0954 & 0 \\
0 & 0.5000
\end{bmatrix}
\end{aligned}
\tag{5.7}
$$

6. Conclusion

In this paper, we propose a general theoretical framework for modeling complex networks with time-invariant topology, composed of nodes with linear internal dynamics and with linear interactions between them.

Nodes perform heterogeneous higher-order internal dynamics, with multi-dimensional input and output vectors. The proposed framework allows to independently define each dynamic interaction between the nodes. Proper notations have been introduced for network topology and the internal dynamics of nodes. The external processes that influence the network dynamics are included in the proposed framework. The analytic solution for the network dynamics is provided in the discrete-time domain, continuous-time domain and the Laplace domain.

The assumption about linear processes on networks allows scalability of the proposed model to large-scale networks and preserves network information. Finally, the reversible hierarchical structuring of complex networks with linear processes is introduced.

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A. List of Notations

Table A.1. Notations for the graph G and DLSS models

Notation	Explanation
G	Graph
$\mathscr{N}$	Set of N nodes of graph G
$\mathscr{L}$	Set of L links of graph G
N	Number of nodes in graph G
L	Number of links in graph G
W	Adjacency matrix of graph G
A_i	State matrix of a DLSS model of node/system i
B_i	Input matrix of a DLSS model of node/system i
C_i	Output matrix of a DLSS model of node/system i
D_i	Feedforward matrix of a DLSS model of node/system i
A_d	Diagonal block matrix composed of A_i matrices, $i \in \mathscr{N}$
B_d	Diagonal block matrix composed of B_i matrices, $i \in \mathscr{N}$
C_d	Diagonal block matrix composed of C_i matrices, $i \in \mathscr{N}$
D_d	Diagonal block matrix composed of D_i matrices, $i \in \mathscr{N}$
A_e	State matrix of a DLSS model of the network
B_e	Input matrix of a DLSS model of the network
C_e	Output matrix of a DLSS model of the network
D_e	Feedforward matrix of a DLSS model of the network

Table A.2. Notations for the links in G_e

Notation	Explanation
l_w	Vector with number of internal links connected to each internal node in G_e
l_ϕ	Vector with number of input links connected to each internal node in G_e
l_ψ	Vector with number of output links connected to each output node in G_e
l_z	Vector with number of external links connected to each output node in G_e
L_w	Total number of internal links in G_e
L_ϕ	Total number of input links in G_e
L_ψ	Total number of output links in G_e
L_z	Total number of external links in G_e
s_w	Vector with number of components of each internal link in G_e
s_ϕ	Vector with number of components of each input link in G_e
s_ψ	Vector with number of components of each output link in G_e
s_z	Vector with number of components of each external link in G_e
S_w	Total number of components of all internal links in G_e
S_ϕ	Total number of components of all input links in G_e
S_ψ	Total number of components of all output links in G_e
S_z	Total number of components of all external links in G_e

Table A.3. Notations for the processes in G_e

Notation	**Explanation**
k	Discrete time variable
t	Continuous time variable
s	Complex variable
n_i	Number of states of i-th node/system in G
n	Vector with number of states of each node/system in G
m_i	Dimension of the input vector u_i of the i-th node/system in G
m	Vector with dimension of the input vector u_i of each node/system in G ($i \in \mathscr{N}$)
p_i	Dimension of the output vector y_i of the i-th node/system in G
p	Vector with dimension of the output vector y_i of each node/system in G ($i \in \mathscr{N}$)
x_i	State vector of the i-th node/system in G
x_e	State vector of entire network
$X_e(s)$	Laplace transform of the state vector $x_e(t)$
u_i	Input vector of the i-th node/system in G
u_d	Aggregated input vectors u_i of each node/system in G ($i \in \mathscr{N}$)
$U_d(s)$	Laplace transform of the aggregated input vector $u_d(t)$
y_i	Output vector of the i-th node/system in G
y_d	Aggregated output vectors y_i of each node/system in G ($i \in \mathscr{N}$)
$Y_d(s)$	Laplace transform of the aggregated output vector $y_d(t)$
$\mathscr{M}$	Set of input nodes in G_e
r	Number of input nodes in G_e
μ_i	Dimension of the i-th external input vector η_i
μ	Vector with dimension of the external input vector η_i of each input node in G_e ($i \in \mathscr{M}$)
M	Sum of elements of the vector μ
η_i	The i-th external input vector in G_e
$H_i(s)$	Laplace transform of the i-th external input vector $\eta_i(t)$
η	Aggregated external input vector
$H(s)$	Laplace transform of the aggregated external input vector $\eta(t)$
$\mathscr{P}$	Set of output nodes in G_e
q	Number of output nodes in G_e
ρ_i	Dimension of the i-th external output vector ξ_i in G_e
ρ	Vector with dimension of the external output vector ξ_i of each input node in G_e ($i \in \mathscr{P}$)
P	Sum of elements of the vector ρ
ξ_i	The i-th external output vector in G_e
$\Xi_i(s)$	Laplace transform of the i-th external output vector $\xi_i(t)$
ξ	Aggregated external output vector
$\Xi(s)$	Laplace transform of the aggregated external output vector $\xi(t)$
$H_i(s)$	Matrix of transfer functions of the i-th node/system in G
$G_d(s)$	Diagonal matrix composed of matrices $H_i(s)$ of each node/system in G ($i \in \mathscr{N}$)
$G_e(s)$	Matrix of transfer functions of the entire network

Table A.4. Notations for the extended graph G_e

Notation	Explanation
G_e	Extended graph
$\mathscr{N}_e$	Set of N_e nodes of extended graph G_e
$\mathscr{L}_e$	Set of L_e links of extended graph G_e
N_e	Number of nodes in extended graph G_e
L_e	Number of links in extended graph G_e
W_e	Adjacency matrix of extended graph G_e
Λ	Incidence matrix of extended graph G_e
Γ	Transposed incidence matrix Λ with all negative entries set to 0
Γ_w	Internal sub-matrix of Γ
Γ_ϕ	Input sub-matrix of Γ
Γ_ψ	Output sub-matrix of Γ
Γ_z	External sub-matrix of Γ
Φ	Matrix that defines the input links existence
Ψ	Matrix that defines the output links existence
Z	Matrix that defines the external links existence
F	Extension of the matrix Γ for higher-dimensional vectors in G_e
F_w	Internal topology matrix, defined upon Γ_w
F_ϕ	Input topology matrix, defined upon Γ_ϕ
F_ψ	Output topology matrix, defined upon Γ_ψ
F_z	External topology matrix, defined upon Γ_z

B. Elaboration of Definition 1

We recall the definition of the matrix Γ:

$$\Gamma = \begin{bmatrix} (\Gamma_\phi)_{(L_w+L_\phi)\times r} & (\Gamma_w)_{(L_w+L_\phi)\times N} & O_{(L_w+L_\phi)\times q} \\ (\Gamma_z)_{(L_\psi+L_z)\times r} & (\Gamma_\psi)_{(L_\psi+L_z)\times N} & O_{(L_\psi+L_z)\times q} \end{bmatrix}$$

Matrix Γ preserves information of the source node of each link in G_e. Each row of the matrix Γ contains exactly one non-zero element and this element is equal to 1.

When $\left(\Gamma_w\right)_{ij} = 1$, it means that j-th internal node provides the i-th link of G_e. In case $\left(\Gamma_\phi\right)_{ij} = 1$, we conclude that the i-th link of G_e originates from the j-th input node. The links connected to the internal nodes are defined with the matrices Γ_w and Γ_ϕ. There are $L_w + L_\phi$ such links (*i.e. internal and input links*).

Remaining $L_\psi + L_z$ links of G_e are connected to the output nodes and they are defined by the matrices Γ_ψ and Γ_z (*i.e. output and external links*). For $\left(\Gamma_\psi\right)_{ij} = 1$, we conclude that the $(L_w + L_\phi + i)$-th link of G_e originates from the j-th internal node. Analogously, $\left(\Gamma_z\right)_{ij} = 1$ indicates that the j-th input node provides the $(L_w + L_\phi + i)$-th link of G_e.

In case all the links in G_e are one-dimensional, i.e. $p_i = 1$ and $\mu_j = 1$, where $i \in \mathscr{N}$, $j \in \mathscr{M}$, the following relations hold:

$$\begin{cases} u_d[k] & = \Gamma_w \cdot y_d[k] + \Gamma_\phi \cdot \eta[k] \\ \xi[k] & = \Gamma_\psi \cdot y_d[k] + \Gamma_z \cdot \eta[k] \end{cases}$$

The definitions of the matrices F_w, F_ϕ, F_ψ and F_z represent the generalization of the matrices Γ_w, Γ_ϕ, Γ_ψ and Γ_z, respectively, in case when not all the links in G_e are one-dimensional.

C. Proof of Theorem 1

After substituting the first relation from (4.13) into the second relation from (4.11) we obtain:

$$y_d[k] = C_d \cdot x_e[k] + D_d \cdot F_w \cdot y_d[k] + D_d \cdot F_\phi \cdot \eta[k]$$

Under the assumption $\det(I - D_d \cdot F_w)^{-1} \neq 0$, we further obtain:

$$y_d[k] = \left(I - D_d \cdot F_w\right)^{-1} \cdot C_d \cdot x_e[k] + \left(I - D_d \cdot F_w\right)^{-1} \cdot (D_d \cdot F_\phi) \cdot \eta[k] \tag{A.1}$$

After substituting relation (A.1) into first relation from (4.13), we obtain the expression for the aggregated input vector u_d:

$$u_d[k] = F_w \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot C_d \cdot x_e[k] + \left(F_w \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot (D_d \cdot F_\phi) + F_\phi\right) \cdot \eta[k] \tag{A.2}$$

Further, after substituting relation (A.2) into first relation from (4.11), we obtain:

$$x_e[k+1] = \left(A_d + B_d \cdot F_w \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot C_d\right) \cdot x_e[k] + \left(B_d \cdot F_w \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot D_d \cdot F_\phi + B_d \cdot F_\phi\right) \cdot \eta[k]$$

from where we recognize the matrices A_e and B_e:

$$\begin{cases} A_e & = A_d + (B_d \cdot F_w) \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot C_d \\ B_e & = (B_d \cdot F_w) \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot (D_d \cdot F_\phi) + B_d \cdot F_\phi \end{cases}$$

Finally, after substituting expression for the aggregated output vector y_d from (A.1) into second relation from (4.13), we obtain:

$$\xi[k] = F_\psi \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot C_d \cdot x_e[k] + F_\psi \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot D_d \cdot F_\phi \cdot \eta[k] + F_z \cdot \eta[k]$$

Hence, we find:

$$\begin{cases} C_e & = F_\psi \cdot (I - D_d \cdot F_w)^{-1} \cdot C_d \\ D_e & = F_\psi \cdot (I - D_d \cdot F_w)^{-1} \cdot (D_d \cdot F_\phi) + F_z \end{cases}$$

which completes the proof. □

D. Homogeneous network with identical interactions between the nodes/systems

In the following subsection, we examine the simplest network with linear processes, i.e. a homogeneous network (a network with nodes that perform identical internal dynamics) with identical dynamic interactions between the nodes/systems. Consequently, dimensions of the external input (2.8) and external output (2.10) vectors, as well of the input (2.6) and output vectors (2.7) are the same:

$$m_i = p_j = \mu_l = \rho_v = p_1 \quad i, j \in \mathscr{N} \quad l \in \mathscr{M} \quad v \in \mathscr{P} \tag{A.1}$$

Node/system i performs internal dynamics defined by (2.4), where the $n_i \times n_i$ state matrix A, the $n_i \times m_i$ input matrix B, the $p_i \times n_i$ output matrix C and the $p_i \times m_i$ feed-forward matrix D are identical for each node/system in the network. For identical interactions, instead of stacking incoming vectors of a certain node into an input/external output vector as in (2.16), they are summed:

$$\begin{aligned} u_i[k] &= \sum_{j \in \mathcal{N}, w_{ji}=1} y_j[k] + \sum_{l \in \mathcal{M}, \phi_{li}=1} \eta_l[k] \\ \xi_i[k] &= \sum_{j \in \mathcal{N}, \psi_{ji}=1} y_j[k] + \sum_{l \in \mathcal{M}, z_{li}=1} \eta_l[k] \end{aligned} \tag{A.2}$$

Therefore, Definition 4.1 for a homogeneous network with identical interactions reduces to:

$$\begin{cases} u_d[k] &= \left(W^T \otimes I_{p_1 \times p_1}\right) \cdot y_d[k] + \left(\Phi^T \otimes I_{p_1 \times p_1}\right) \cdot \eta[k] \\ \xi[k] &= \left(\Psi^T \otimes I_{p_1 \times p_1}\right) \cdot y_d[k] + \left(Z^T \otimes I_{p_1 \times p_1}\right) \cdot \eta[k] \end{cases} \tag{A.3}$$

Analogously to the Theorem 5.1, we provide the parameters of the DLSS model for the time dynamics of the entire network:

$$\begin{cases} A_e = \left(W^T \otimes B\right) \cdot \left(I_{Np_1 \times Np_1} - W^T \otimes D\right)^{-1} \cdot \left(I_N \otimes C\right) + \left(I_{N \times N} \otimes A\right) \\ B_e = \left(W^T \otimes B\right) \cdot \left(I_{Np_1 \times Np_1} - W^T \otimes D\right)^{-1} \cdot \left(\Phi^T \otimes D\right) + \left(\Phi^T \otimes B\right) \\ C_e = \left(\Psi^T \otimes I_{p_1 \times p_1}\right) \cdot \left(I_{Np_1 \times Np_1} - W^T \otimes D\right)^{-1} \cdot \left(I_{N \times N} \otimes C\right) \\ D_e = \left(\Psi^T \otimes I_{p_1 \times p_1}\right) \cdot \left(I_{Np_1 \times Np_1} - W^T \otimes D\right)^{-1} \cdot \left(\Phi^T \otimes D\right) + \left(Z^T \otimes I_{p_1 \times p_1}\right) \end{cases} \tag{A.4}$$

while, in the case, the $p_1 \times p_1$ feed-forward matrix $D = O_{p_1 \times p_1}$, the solution for parameters of the governing model (5.2) becomes considerably simpler:

$$\begin{cases} A_e = \left(W^T \otimes B \cdot C\right) + \left(I_{N \times N} \otimes A\right) \\ B_e = \left(\Phi^T \otimes B\right) \\ C_e = \left(\Psi^T \otimes C\right) \\ D_e = \left(Z^T \otimes I_{p_1 \times p_1}\right) \end{cases} \tag{A.5}$$

E. Continuous-time linear processes on complex networks

E.1 *Time-domain analysis*

The continuous-time linear dynamics of the i-th node/system of the network obey a similar governing equation as (2.4):

$$\begin{cases} \frac{dx_i(t)}{dt} &= A_i \cdot x_i(t) + B_i \cdot u_i(t) \\ y_i(t) &= C_i \cdot x_i(t) + D_i \cdot u_i(t) \end{cases} \tag{A.1}$$

where t denotes continuous time. We revise the definition of the $\sum_{i=1}^{N} m_i \times 1$ aggregated input vector u_d from (4.12), the $\sum_{i=1}^{N} p_i \times 1$ aggregated output vector y_d from (4.12), the $M \times 1$ aggregated external input vector η from (2.9) and the $P \times 1$ aggregated external output vector ξ from (2.11) as follows:

$$\begin{aligned} u_d(t) &= \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{bmatrix} \quad y_d(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_N(t) \end{bmatrix} \\ \eta(t) &= \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \vdots \\ \eta_r(t) \end{bmatrix} \quad \xi(t) = \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \\ \vdots \\ \xi_q(t) \end{bmatrix} \end{aligned} \tag{A.2}$$

The aim is to determine the dynamics between the aggregated external output vector $\xi(t)$ and the aggregated external input vector $\eta(t)$, by following governing equations:

$$\begin{aligned} \frac{dx_e(t)}{dt} &= A_e \cdot x_e(t) + B_e \cdot \eta(t) \\ \xi(t) &= C_e \cdot x_e(t) + D_e \cdot \eta(t) \end{aligned} \tag{A.3}$$

where the $\sum_{i=1}^{N} n_i$ state vector $x_e(t)$ is defined as follows:

$$x_e(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix} \tag{A.4}$$

The direct continuous-time analogy of Theorem 1 in discrete-time domain is as follows:

THEOREM A.1 The matrices A_e, B_e, C_e and D_e from the DLSS equations in (A.3),

$$\begin{aligned} \frac{dx_e(t)}{dt} &= A_e \cdot x_e(t) + B_e \cdot \eta(t) \\ \xi(t) &= C_e \cdot x_e(t) + D_e \cdot \eta(t) \end{aligned}$$

provided the matrix $(I - D_d \cdot F_w)$ is non-singular or $(D_d \cdot F_w)$ has not an eigenvalue 1, are explicitly determined as follows:

$$\begin{cases} A_e = (B_d \cdot F_w) \cdot (I - D_d \cdot F_w)^{-1} \cdot C_d + A_d \\ B_e = (B_d \cdot F_w) \cdot (I - D_d \cdot F_w)^{-1} \cdot (D_d \cdot F_\phi) + B_d \cdot F_\phi \\ C_e = F_\psi \cdot (I - D_d \cdot F_w)^{-1} \cdot C_d \\ D_e = F_\psi \cdot (I - D_d \cdot F_w)^{-1} \cdot (D_d \cdot F_\phi) + F_z \end{cases} \tag{A.5}$$

while Corollary 1 remains the same.

E.2 *Laplace-domain analysis*

The unilateral (one-sided) Laplace transform, denoted as $\mathscr{L}\{f(t)\}$, of a continuous-time function $f(t)$ that is defined for all real numbers $t \geqslant 0$ is the complex function $F(s)$ defined as follows:

$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt \tag{A.6}$$

where s is a complex variable. In case the function $f(t)$ is defined also for negative real numbers, the bilateral (two-sided) Laplace transform is defined as en extension of (A.6), where the limits of the integral become entire real axis:

$$F(s) = \mathscr{L}\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t) dt \tag{A.7}$$

The inverse Laplace transform, denoted as $\mathscr{L}^{-1}\{F(s)\}$ is defined by:

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i} \lim_{T\to\infty} \oint_{\gamma-iT}^{\gamma+iT} e^{st} F(s)\, ds \tag{A.8}$$

where the real number γ defines the contour path of integration, that belongs to the region of convergence of $F(s)$.

The governing equations of the i-th node/system in continuous-time domain from (A.1) can be transformed into transfer functions using Laplace transform:

$$Y_i(s) = G_i(s) \cdot U_i(s) = \Big(C_i \cdot (sI - A_i)^{-1} \cdot B_i + D_i \Big) \cdot U_i(s) \tag{A.9}$$

where the $p_i \times 1$ complex output vector $Y_i(s)$ and the $m_i \times 1$ complex input vector $U_i(s)$ are the Laplace transforms of the output vector $y_i(t)$ and the input vector $u_i(t)$, respectively. The $p_i \times m_i$ complex matrix $G_i(s)$ is a matrix of transfer functions between the complex vectors $Y_d(s)$ and $U_d(s)$, where the $\big(G_i(s)\big)_{jk}$ transfer function defines the dynamics between the j-th component of the complex output vector $\big(Y_i(s)\big)_j$ and the k-th component of the complex input vector $\big(U_i(s)\big)_k$.

The Laplace transforms of the aggregated input vector $u_d(t)$, aggregated output vector $y_d(t)$, aggregated external input vector $\eta(t)$ and aggregated external output vector $\xi(t)$ from (A.2) are defined as follows, respectively:

$$\begin{aligned} U_d(s) &= \begin{bmatrix} U_1(s) \\ U_2(s) \\ \vdots \\ U_N(s) \end{bmatrix} \quad Y_d(s) = \begin{bmatrix} Y_1(s) \\ Y_2(s) \\ \vdots \\ Y_N(s) \end{bmatrix} \\ H(s) &= \begin{bmatrix} H_1(s) \\ H_2(s) \\ \vdots \\ H_r(s) \end{bmatrix} \quad \Xi(s) = \begin{bmatrix} \Xi_1(s) \\ \Xi_2(s) \\ \vdots \\ \Xi_q(s) \end{bmatrix} \end{aligned} \tag{A.10}$$

By defining the $\sum_{i=1}^N p_i \times \sum_{i=1}^N m_i$ complex matrix $G_d(s)$ as a block diagonal matrix, composed of the transfer functions $G_i(s)$ of each individual node/system (i.e. $i \in \mathscr{N}$):

$$G_d(s) = \text{diagonal}\begin{bmatrix} G_1(s) & G_2(s) & \dots & G_N(s) \end{bmatrix} = C_d \cdot (sI - A_d)^{-1} \cdot B_d + D_d \tag{A.11}$$

we are able to define the dynamics between the complex aggregated output vector $Y_d(s)$ and complex aggregated input vector $U_d(s)$ in a compact form:

$$Y_d(s) = G_d(s) \cdot U_d(s) \tag{A.12}$$

The aim of this subsection is to determine the $P \times M$ complex matrix $G_e(s)$ of transfer functions between the complex aggregated external output vector $\Xi(s)$ and the complex aggregated external input vector $H(s)$:

$$\Xi(s) = G_e(s) \cdot H(s) \tag{A.13}$$

where the $P \times 1$ complex aggregated external output vector $\Xi(s)$ and the $M \times 1$ complex aggregated external input vector $H(s)$ are Laplace transforms of the aggregated external input vector $\xi(t)$ and the aggregated external input vector $\eta(t)$, respectively.

The Laplace transform of the direct continuous-time analogy of Definition 1 in discrete-time domain is as follows:

$$\begin{cases} U_d(s) & = F_w \cdot Y_d(s) + F_\phi \cdot H(s) \\ \Xi(s) & = F_\psi \cdot Y_d(s) + F_z \cdot H(s) \end{cases} \tag{A.14}$$

THEOREM A.2 The complex matrix $G_e(s)$ of transfer functions from (A.13) is explicitly determined as follows:

$$G_e(s) = F_\psi \cdot \Big(I - G_d(s) \cdot F_w\Big)^{-1} \cdot G_d(s) \cdot F_\phi + F_z \tag{A.15}$$

Proof. We provide two different proofs of the Theorem A.2. The first proof is based upon (A.14).

1) After substituting first relation from (A.14) into (A.12), we obtain:

$$Y_d(s) = G_d(s) \cdot F_w \cdot Y_d(s) + G_d(s) \cdot F_\phi \cdot H(s)$$

from where, under the assumption $\det(I - F_w \cdot G_d(s)) \neq 0$ we express the complex aggregated output vector $Y_d(s)$:

$$Y_d(s) = \Big(I - G_d(s) \cdot F_w\Big)^{-1} \cdot G_d(s) \cdot F_\phi \cdot H(s) \tag{A.16}$$

Next, we substitute (A.16) into second relation from (A.14) and obtain:

$$\Xi(s) = \Big(F_\psi \cdot \big(I - G_d(s) \cdot F_w\big)^{-1} \cdot G_d(s) \cdot F_\phi + F_z\Big) \cdot H(s)$$

which completes the proof.

2) In Theorem 2, the dynamics between the aggregated external output vector $\xi(t)$ and the aggregated external input vector $\eta(t)$ are determined by the governing equations in (A.3). Hence, the Laplace transform of the governing equations from (A.3) is actually the complex matrix $G_e(s)$ of transfer functions between the complex aggregated external output vector $\Xi(s)$ and the complex aggregated external input vector $H(s)$:

$$G_e(s) = C_e \cdot \big(sI - A_e\big)^{-1} \cdot B_e + D_e \tag{A.17}$$

After substituting (A.5) into (A.17), we obtain:

$$\begin{aligned} G_e(s) =& F_\psi \cdot \big(I - D_d \cdot F_w\big)^{-1} \cdot C_d \cdot \Big(sI - \big(B_d \cdot F_w\big) \cdot \big(I - D_d \cdot F_w\big)^{-1} \cdot C_d - A_d\Big)^{-1} \\ & \cdot \Big((B_d \cdot F_w) \cdot \big(I - D_d \cdot F_w\big)^{-1} \cdot \big(D_d \cdot F_\phi\big) + B_d \cdot F_\phi\Big) \\ & + F_\psi \cdot \big(I - D_d \cdot F_w\big)^{-1} \cdot \big(D_d \cdot F_\phi\big) + F_z \end{aligned} \tag{A.18}$$

We right multiply the inverse term $(sI-A_e)^{-1}$ from (A.18) with $(sI-A_d)\cdot(sI-A_d)^{-1}$ (i.e. with identity matrix) and regroup the terms inside the same term:

$$\begin{aligned} G_e(s) =&F_\psi \cdot \big(I-D_d\cdot F_w\big)^{-1}\cdot C_d \cdot \Big(\big(sI-A_d\big) - \big(B_d\cdot F_w\big)\cdot\big(I-D_d\cdot F_w\big)^{-1}\cdot C_d\Big)^{-1} \\ &\cdot \big(sI-A_d\big)\cdot\big(sI-A_d\big)^{-1}\cdot\Big(\big(B_d\cdot F_w\big)\cdot\big(I-D_d\cdot F_w\big)^{-1}\cdot\big(D_d\cdot F_\phi\big)+B_d\cdot F_\phi\Big) \\ &+F_\psi\cdot\big(I-D_d\cdot F_w\big)^{-1}\cdot\big(D_d\cdot F_\phi\big)+F_z \end{aligned} \tag{A.19}$$

After applying the property of a matrix inverse onto the product $(sI-A_e)^{-1}\cdot\big((sI-A_d)^{-1}\big)^{-1}$ from (A.19) we obtain:

$$\begin{aligned} G_e(s) =&F_\psi \cdot \big(I-D_d\cdot F_w\big)^{-1}\cdot C_d \cdot \Big(I-\big(sI-A_d\big)^{-1}\cdot\big(B_d\cdot F_w\big)\cdot\big(I-D_d\cdot F_w\big)^{-1}\cdot C_d\Big)^{-1} \\ &\cdot \Big(\big(sI-A_d\big)^{-1}\cdot\big(B_d\cdot F_w\big)\cdot\big(I-D_d\cdot F_w\big)^{-1}\cdot\big(D_d\cdot F_\phi\big)+\big(sI-A_d\big)^{-1}\cdot\big(B_d\cdot F_\phi\big)\Big) \\ &+F_\psi\cdot\big(I-D_d\cdot F_w\big)^{-1}\cdot\big(D_d\cdot F_\phi\big)+F_z \end{aligned} \tag{A.20}$$

We define the $\sum_{i=1}^N n_i \times \sum_{i=1}^N p_i$ complex matrix $K(s)$ as follows:

$$K(s) = \big(sI-A_d\big)^{-1}\cdot\big(B_d\cdot F_w\big)\cdot\big(I-D_d\cdot F_w\big)^{-1} \tag{A.21}$$

and observe the following matrix product from (A.20):

$$C_d\cdot\big(I-K(s)\cdot C_d\big)^{-1} \tag{A.22}$$

We claim the next identity holds:

$$C_d\cdot\big(I-K(s)\cdot C_d\big)^{-1} = \big(I-C_d\cdot K(s)\big)^{-1}\cdot C_d \tag{A.23}$$

where the identity matrix I from the left-hand side is of dimensions $\sum_{i=1}^N n_i \times \sum_{i=1}^N n_i$ and the identity matrix from the right-hand side of (A.23) has dimensions $\sum_{i=1}^N p_i \times \sum_{i=1}^N p_i$. We prove (A.23) by contradiction.

We denote the difference between the left-hand and the right-hand side of (A.23) as a complex matrix $E(s)$ of dimensions $\sum_{i=1}^N p_i \times \sum_{i=1}^N n_i$:

$$E(s) = C_d\cdot\big(I-K(s)\cdot C_d\big)^{-1} - \big(I-C_d\cdot K(s)\big)^{-1}\cdot C_d \tag{A.24}$$

After left multiplying with $\big(I-C_d\cdot K(s)\big)$ and right multiplying with $\big(I-K(s)\cdot C_d\big)$ both sides of (A.24) we obtain:

$$\big(I-C_d\cdot K(s)\big)\cdot E(s)\cdot\big(I-K(s)\cdot C_d\big) = O$$

Left side of last equation is always zero, thus we conclude $E(s)=O$. We import the proven identity (A.23) into (A.20) and obtain:

$$\begin{aligned} G_e(s) =&F_\psi \cdot \big(I-D_d\cdot F_w\big)^{-1}\cdot \Big(I-C_d\cdot\big(sI-A_d\big)^{-1}\cdot\big(B_d\cdot F_w\big)\cdot\big(I-D_d\cdot F_w\big)^{-1}\Big)^{-1} \\ &\cdot \Big(C_d\cdot\big(sI-A_d\big)^{-1}\cdot\big(B_d\cdot F_w\big)\cdot\big(I-D_d\cdot F_w\big)^{-1}\cdot\big(D_d\cdot F_\phi\big) \\ &+C_d\cdot\big(sI-A_d\big)^{-1}\cdot\big(B_d\cdot F_\phi\big)\Big)+F_\psi\cdot\big(I-D_d\cdot F_w\big)^{-1}\cdot\big(D_d\cdot F_\phi\big)+F_z \end{aligned} \tag{A.25}$$

We regroup the terms inside the fourth product term of (A.25) in such a way to build a matrix, whose inverse appears as the third product term in (A.25):

$$\begin{aligned}G_e(s) =&F_\psi \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot \left(I - C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_w\right) \cdot \left(I - D_d \cdot F_w\right)^{-1}\right)^{-1} \\ &\cdot \Bigg(-\left(I - C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_w\right) \cdot \left(I - D_d \cdot F_w\right)^{-1}\right) \cdot \left(D_d \cdot F_\phi\right) \\ &+ \left(D_d \cdot F_\phi\right) + C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_\phi\right)\Bigg) \\ &+ F_\psi \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot \left(D_d \cdot F_\phi\right) + F_z\end{aligned} \tag{A.26}$$

After multiplying the third and the fourth product terms from (A.26), we obtain:

$$\begin{aligned}G_e(s) =& -F_\psi \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot \left(D_d \cdot F_\phi\right) + F_\psi \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot \\ &\left(I - C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_w\right) \cdot \left(I - D_d \cdot F_w\right)^{-1}\right)^{-1} \\ &\cdot \left(C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_\phi\right) + \left(D_d \cdot F_\phi\right)\right) \\ &+ F_\psi \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot \left(D_d \cdot F_\phi\right) + F_z\end{aligned} \tag{A.27}$$

The first and the third sum terms from (A.27) are the same, but with opposite signs. Hence, we obtain:

$$\begin{aligned}G_e(s) =&F_\psi \cdot \left(I - D_d \cdot F_w\right)^{-1} \cdot \left(I - C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_w\right) \cdot \left(I - D_d \cdot F_w\right)^{-1}\right)^{-1} \\ &\cdot \left(C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_\phi\right) + \left(D_d \cdot F_\phi\right)\right) + F_z\end{aligned} \tag{A.28}$$

Finally, after applying the property of a matrix inverse onto the product of the second and third product terms in (A.28), we obtain the final form for $G_e(s)$:

$$\begin{aligned}G_e(s) =&F_\psi \cdot \left(I - \left(C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_w\right) + D_d \cdot F_w\right)\right)^{-1} \\ &\cdot \left(C_d \cdot \left(sI - A_d\right)^{-1} \cdot \left(B_d \cdot F_\phi\right) + D_d \cdot F_\phi\right) + F_z\end{aligned} \tag{A.29}$$

which equals (A.15) and completes the proof. □

F. Intuitive explanation of Theorem 1

In order to provide an intuitive explanation of Theorem 1, we propose a graph representation of the network dynamics. Throughout the explanation, we use simple networks to support our observations. Upon these examples and provided observations, we induce the solution for the network dynamics, provided in Theorem 1.

F.1 *Graph representation of the DLSS process*

A DLSS process can be represented as a graph, whose nodes model values of the process variables in discrete-time k, while these variables are scaled and transmitted over links of the graph.

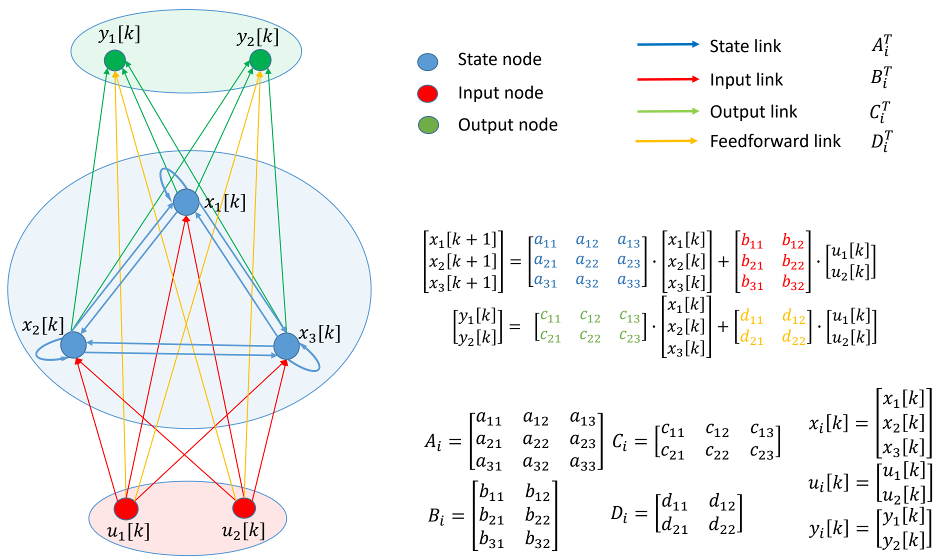

FIG. A.10. A graph representation of the DLSS process inside the i-th system of a network, where $n_i = 3$, $m_i = 2$, $p_i = 2$

Consider the node/system i of the network. The state vector x_i is of dimension $n_i \times 1$. We define n_i nodes that model the value of the state vector x_i, where the j-th node contains the value of the j-th state $\left(x_i\right)_j$ at discrete time k. We refer to these nodes as *state nodes* (blue nodes in Fig A.10). The state nodes are strongly connected via *state links* (blue links in Fig A.10), whose weights are defined by the $n_i \times n_i$ state matrix $\left(A_i\right)^T$.

The input vector u_i of the system i has dimension $m_i \times 1$. We define m_i additional nodes and refer to them as *input nodes* (red nodes in Fig A.10). The input nodes represent the value of the input vector u_i at discrete time k, where the j-th input node represents the value of the j-th component of the input vector $\left(u_i\right)_j$. An input node is connected to each state node by *input links* (red links in Fig A.10), i.e. input links connect the input nodes to the state nodes. The weights of the input links are defined by the $m_i \times n_i$ input matrix $\left(B_i\right)^T$.

The $p_i \times 1$ output vector y_i is modelled by p_i *output nodes* (green nodes in Fig A.10), containing the value of the vector y_i at discrete time k. A state node is connected to each output node by the *output links* (green links in Fig A.10). The weights of the output links are defined by the $n_i \times p_i$ output matrix $\left(C_i\right)^T$.

Finally, an input node is connected to each output node via *feedforward links* (yellow links in Fig A.10), whose weights are defined by the $m_i \times p_i$ feedforward matrix $\left(D_i\right)^T$. We sketch the graph representation of the DLSS process inside the system i in Fig A.10, where $n_i = 3$, $m_i = 2$ and $p_i = 2$.

The process variables are transmitted over defined four types of links. The weight of a link defines how the signal is scaled while being transmitted over that link. While transmitting the signal, **state and input links impose fixed time delay** equal to the one discrete-time instant. On the other side, **output and feedforward links do not impose time delay**, i.e. they **instantaneously transmit the signal**.

From the graph representation of the DLSS process, we provide relation between the state vector x_i

and the input vector u_i of the i-th system from the network:

$$x_i[k] = A_i \cdot x_i[k-1] + B_i \cdot u_i[k-1]$$

where the previous values of the state vector x_i and of the input vector u_i appear because both the state links, whose weights are defined by the state matrix $\left(A_i\right)^T$ and the input links, whose weights are defined by the input matrix $\left(B_i\right)^T$ impose time delay. After substituting $k \to k+1$ in previous equation, we obtain:

$$x_i[k+1] = A_i \cdot x_i[k] + B_i \cdot u_i[k]$$

Analogously, we provide relation between the output vector y_i, the state vector x_i and the input vector u_i, using graph representation of the DLSS process:

$$y_i[k] = C_i \cdot x_i[k] + D_i \cdot u_i[k]$$

where there is no time delay, because both the output links, whose weights are defined by the output matrix $\left(C_i\right)^T$ and the feedforward links, whose weights are defined by the feedforward matrix $\left(D_i\right)^T$ do not impose time delay, i.e. they transmit the signal instantaneously. By merging last two equations, we obtain the governing equations for the DLSS process of the system i from the network:

$$\begin{aligned} x_i[k+1] &= A_i \cdot x_i[k] + B_i \cdot u_i[k] \\ y_i[k] &= C_i \cdot x_i[k] + D_i \cdot u_i[k] \end{aligned}$$

We condense the graph representation of the DLSS process as follows:

- n_i state nodes are condensed into one state node, which contains the value of the state vector x_i at discrete-time k
- m_i input nodes are condensed into one input node, that contains the value of the input vector u_i at discrete-time k
- p_i output nodes are condensed into one output node, containing the value of the output vector y_i at discrete-time k
- The state links are condensed into one state link that represents a self-loop for the condensed state node. This condensed link is defined by the state matrix $\left(A_i\right)^T$
- The input links are condensed into one input link, that connects the condensed input node to the condensed state node and is defined by the input matrix $\left(B_i\right)^T$
- The output links are condensed into one output link, connecting the condensed state node to the condensed output node and is defined by the output matrix $\left(C_i\right)^T$
- The feedforward links are condensed into one feedforward link that connects the condensed input node to the condensed output node and is defined by the feedforward matrix $\left(D_i\right)^T$.

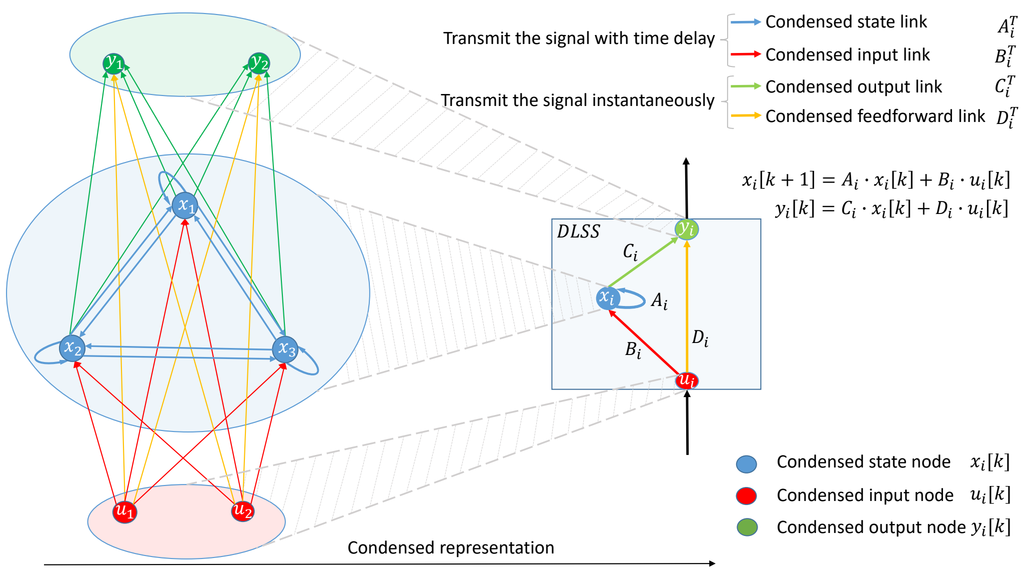

FIG. A.11. A condensed graph representation of the DLSS process from Fig A.10

A condensed graph representation of the DLSS process from Fig A.10 is sketched in Fig A.11.

Following previously defined properties of the links from the graph representation of the DLSS process, we stress that the condensed state and the condensed input link impose the fixed time delay while transmitting the signal, whereas the condensed output and condensed feedforward link instantaneously transmit the signal.

The network dynamics are defined by the DLSS governing equations from (3.1). Thus, the network dynamics can be also presented by a graph, analogously to the graph representation of the DLSS process of individual system from the network. In the graph represetnation of the entire network dynamics, there are $\sum_{i=1}^{N} n_i$ state nodes, M input and P output nodes. Furthermore, there are $\left(\sum_{i=1}^{N} n_i\right)^2$ state links, $M \cdot \sum_{i=1}^{N} n_i$ input, $\sum_{i=1}^{N} n_i \cdot P$ output and $M \cdot P$ feedforward links. The weights of the state links are defined by the $\sum_{i=1}^{N} n_i \times \sum_{i=1}^{N} n_i$ state matrix $\left(A_e\right)^T$, whereas the $\sum_{i=1}^{N} m_i \times \sum_{i=1}^{N} n_i$ input matrix $\left(B_e\right)^T$ determines the weights of the input links. Finally, the $\sum_{i=1}^{N} n_i \times \sum_{i=1}^{N} p_i$ output matrix $\left(C_e\right)^T$ and the $\sum_{i=1}^{N} m_i \times \sum_{i=1}^{N} p_i$ feedforward $\left(D_e\right)^T$ contain the weights of the output and feedforward links, respectively.

Network topology defines the interconnection pattern of the nodes/systems. On the other side, a graph representation of the DLSS process within a node/system defines the dynamics inside that node, i.e. it defines the dynamics locally, without any knoledge of how the dynamics interact with the dynamics within other nodes/systems of the network. Therefore, network dynamics are determined by both the underlying topology and the dynamics of individual nodes/systems.

By combining condensed graph representation of the DLSS process of the individual systems and the underlying topology, we provide a condensed graph representation of the netork dynamics in Fig A.12, for a network of 3 nodes, composing a triangle. Network topology link (black links in Fig A.12) defines the relation between the output vector of the source node/system and input vector of the destination node/system. As an example, a network topology link between the nodes/systems 2 and 1 implies $y_2[k] = u_1[k]$.

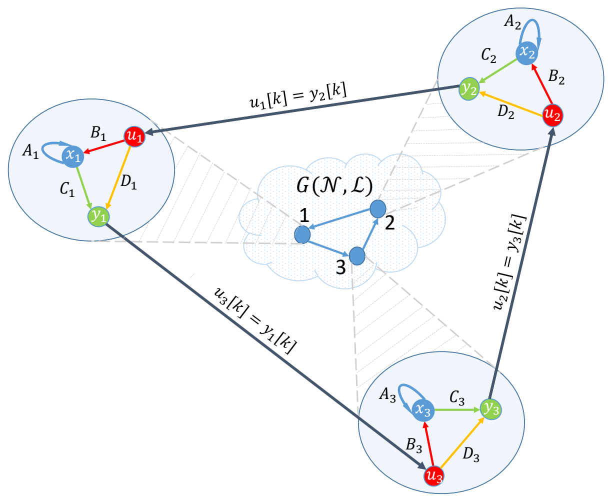

FIG. A.12. A condensed graph representation of the network dynamics, with $N = 3$

F.2 *State matrix A_e*

The $\sum_{i=1}^{N} n_i \times \sum_{i=1}^{N} n_i$ state matrix A_e of the network dynamics is composed of $N \times N$ block elements, where the $n_i \times n_j$ block element $\left(A_e\right)_{ij}$ defines the impact of the state vector x_j onto the state vector x_i. The transposed state matrix A_e^T defines the weights of the state links from the graph representation of the network dynamics. Thus, each state link from the graph representation of the network dynamics imposes fixed time delay while transmitting the signal, equal to the one discrete-time instant.

We first consider the $\left(A_e\right)_{13}$ block element of the network presented in Fig A.12. **Determining the $\left(A_e\right)_{13}$ block element is equivalent to determining the sum of all the paths from the node x_3 to the node x_1, using the condensed graph representation of the network dynamics (Fig A.12), under the constraint that each path imposes fixed time delay, equal to the one discrete-time instant**.

Since each path from node x_3 to node x_1 ends with the condensed input link of the system 1 (input links impose time delay), the remaining part of each path cannot introduce time delay. In this particular case, there is such a path, over the condensed output link of the system 3 and the condensed feedforward link of the system 2, i.e. $(B_1 \cdot D_2 \cdot C_3)$. Moreover, since the condensed feedforward links impose no time delay and the nodes of the network from Fig A.12 compose a ring, there are infinitely many paths that satisfy previous constraint: $\left(B_1 \cdot D_2 \cdot (D_3 \cdot D_1 \cdot D_2)^k \cdot C_3\right), k > 0$. Thus, we conclude:

$$(A_e)_{13} = B_1 \cdot D_2 \cdot \sum_{k=0}^{\infty} (D_3 \cdot D_1 \cdot D_2)^k \cdot C_3$$

We next consider the block element $\left(A_e\right)_{11}$. It is determined as a sum of all the paths that start from and end in the node x_1, where each such a path introduces unity time delay. The condensed state link of the system 1 introduces time delay and creates the self-loop for the node x_1. Hence, it is a valid path. Also, there are infinitely many paths that start with the condensed output link of the system 1, continues with the condensed feedforward link of the systems and finish with the condensed input link of the first system: $(\left(B_1 \cdot D_2 \cdot D_3 \cdot (D_1 \cdot D_2 \cdot D_3)^k \cdot C_1\right), k > 0)$. Thus, we conclude:

$$(A_e)_{11} = A_1 + B_1 \cdot D_2 \cdot D_3 \cdot \sum_{k=0}^{\infty} (D_1 \cdot D_2 \cdot D_3)^k \cdot C_1$$

Remaining block elements of A_e can be determined analogously.

Each node from the network presented in Fig A.12 has the unity in-degree. In such a case, the input vector of a system is the same as the output vector of its neighbour, i.e. if the system j receives the link only from the system i, we have $u_j[k] = y_i[k]$.

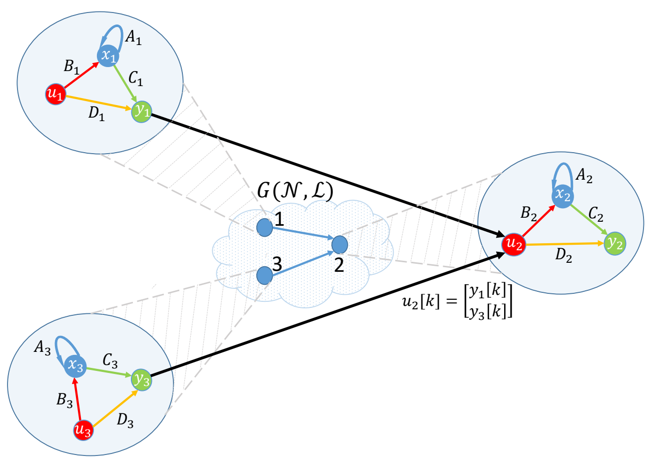

FIG. A.13. A condensed graph representation of the network dynamics, where $N = 3$, with non unity in-degree distribution

We analyse next a case when a node/system has more than one neighbour, which is the case for the system 2 from the network presented in Fig A.13. Nodes 1 and 3 both provide link to the node 2. Hence, the input vector u_2 is composed of the output vectors y_1 and the output vector y_3:

$$\left(u_2[k]\right)_{m_2 \times 1} = \begin{bmatrix} \left(y_1[k]\right)_{p_1 \times 1} \\ \left(y_3[k]\right)_{p_3 \times 1} \end{bmatrix}$$

where the first p_1 components of the input vector u_2 are equal to the output vector y_1, while the remaining p_3 components of the input vector u_2 are equal to the output vector y_3. Furthermore, first p_1 columns of the input matrix B_2 and of the feedforward matrix D_2 regard the output vector y_1, while the remaining p_3 columns of these matrices are related to the output vector y_3.

We consider the block element $\left(A_e\right)_{21}$ for the network presented in Fig A.13, i.e. how the state vector x_1 influences the state vector x_2. From Fig A.13, we conclude that there is only one path that satisfies previously defined properties and this path consists of the condensed output link of the system 1 and the condensed input link of the system 2. Additionally, we need only first p_1 columns of the input matrix B_2, because the output vector y_1 constitutes first p_1 components of the input vector u_2.

Therefore, we introduce the $(S_w + S_\phi) \times P$ matrix F_w, composed of $(L_w + L_\phi) \times N$ block elements, defined as follows:

$$\left(F_w\right)_{ij} = \begin{cases} I_{(s_w+s_\phi)_i} & \text{if } \left(\Gamma_w\right)_{ij} = 1 \\ O_{(s_w+s_\phi)_i \times p_j} & \text{otherwise} \end{cases} \tag{A.1}$$

Each block row of F_w is related to the output vector of a certain system from the network. First $\left(l_w + l_\phi\right)_1$ block rows of F_w (in total $\sum_{k=1}^{(l_w+l_\phi)_1} (s_w + s_\phi)_k = m_1$ rows) regard the output vectors of the systems that are connected to the system 1. Next $\left(l_w + l_\phi\right)_2$ block rows of F_w are related to the output vectors of the systems connected to the system 2 and so on. Last $\left(l_w + l_\phi\right)_N$ block rows of F_w are related to the output vectors of the systems that are connected to the N-th system of the network. Therefore, we can further combine block rows into new N block rows, where the i-th block row regards the output vectors of the systems connected to the system i and is composed of m_i rows. Therefore, we conclude that matrix F_w is composed of $N \times N$ block elements, where the block element $(F_w)_{ij}$ has dimensions $m_i \times p_j$.

We next analyse the matrix product $(B_d \cdot F_w)$, by considering the block element $\left(B_d \cdot F_w\right)_{ij} = B_i \cdot \left(F_w\right)_{ij}$. In case the system j is directly connected to the system i, (i.e. $w_{ij} = 1$), the $m_i \times p_j$ block element $\left(B_d \cdot F_w\right)_{ij}$ contains the p_j columns of the input matrix B_i, that are related to the output vector y_j. Otherwise, when there is no link between the nodes/systems i and j (i.e. when $w_{ij} = 0$) the block element $\left(B_d \cdot F_w\right)_{ij}$ is a zero matrix. The analogous explanation holds for the block element $\left(D_d \cdot F_w\right)_{ij} = D_i \cdot \left(F_w\right)_{ij}$ of a product $(D_d \cdot F_w)$.

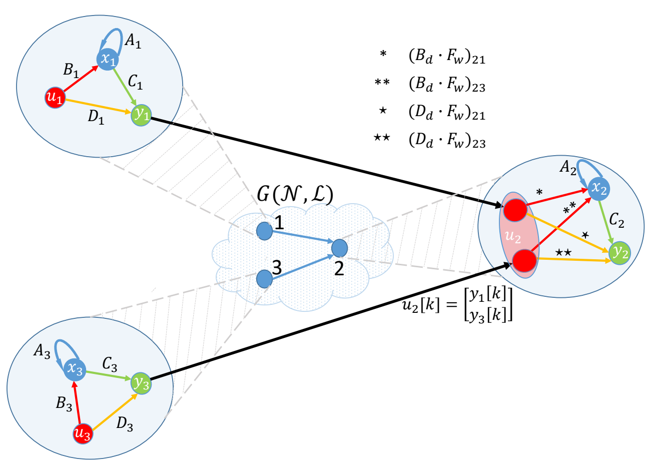

FIG. A.14. Explanation of the products $(B_d \cdot F_w)$ and $(D_d \cdot F_w)$, for a network from Fig A.13

We apply these products to the network from Fig A.13 and sketch it in Fig A.14. Block elements

$\left(B_d \cdot F_w\right)_{21}$ and $\left(B_d \cdot F_w\right)_{23}$ contain the first p_1 and the last p_3 columns of the input matrix B_2, respectively. Analogously, block elements $\left(D_d \cdot F_w\right)_{21}$ and $\left(D_d \cdot F_w\right)_{23}$ split the feedforward matrix D_2 into two sub-matrices. The first sub-matrix $\left(D_d \cdot F_w\right)_{21}$ is composed of the first p_1 columns of the feedforward matrix D_2, that are related to the output vector y_1, while the second sub-matrix $\left(D_d \cdot F_w\right)_{23}$ contains the remaining p_2 columns, related to the output vector y_3. Therefore, state matrix A_e for the network from Fig A.14 is determined as follows:

$$A_e = \begin{bmatrix} A_1 & O_{n_1 \times n_2} & O_{n_1 \times n_3} \\ \left(B_d \cdot F_w\right)_{21} \cdot C_1 & A_2 & \left(B_d \cdot F_w\right)_{23} \cdot C_3 \\ O_{n_3 \times n_1} & O_{n_3 \times n_2} & A_3 \end{bmatrix}$$

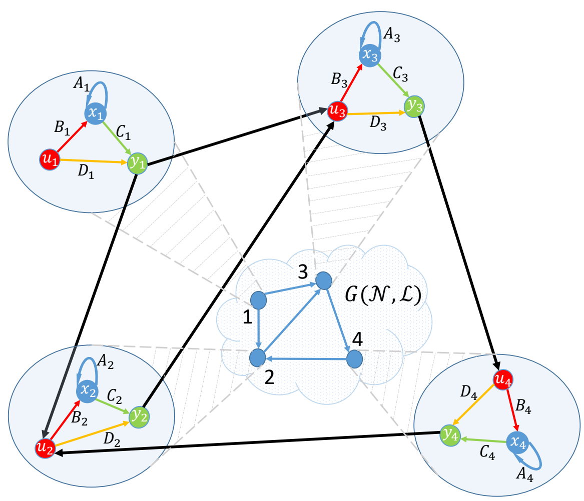

FIG. A.15. A condensed graph representation of the network dynamics, with $N = 4$ nodes and $L = 5$ links

The $\sum_{i=1}^{N} p_i \times \sum_{i=1}^{N} p_i$ matrix $\left(D_d \cdot F_w\right)$ is composed of the $N \times N$ block elements. If there is a direct link from node i to node j (i.e. $w_{ij} = 1$), the $p_i \times p_j$ block element $\left(D_d \cdot F_w\right)_{ij} = D_i \cdot \left(F_w\right)_{ij}$ contains the p_j columns of the feedforward matrix D_i that are related to the output vector y_j, otherwise (i.e $w_{ij} = 0$) it is a zero matrix.

The product $\left(D_d \cdot F_w\right)$ can be used to compute relation between the output vectors y_i and y_j over only feddforward links of all the systems in the network. A merged graph representation of the network dynamics is sketched in Fig A.15, for a network of $N = 4$ nodes.

Since we analyse the relation between the output vectors y_i and y_j over the feedforward links, other three types of links have been neglected in Fig A.16. Furthermore, in Fig A.17 we apply the multiplication of the block diagonal feedforward matrix D_d with the matrix F_w. This multiplication enables us to further simplify condensed graph representation of network dynamics, using only feedforward links, as presented in Fig A.17.

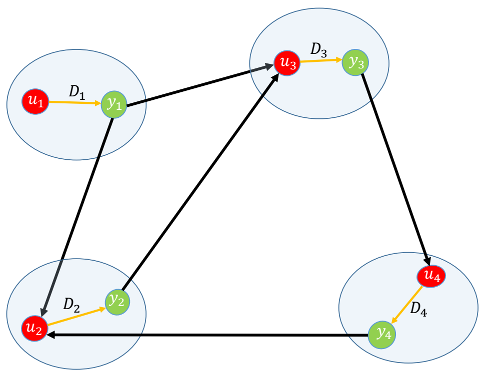

FIG. A.16. Neglected condensed state, input and output links from the graph representation of network dynamics of the network from Fig A.15

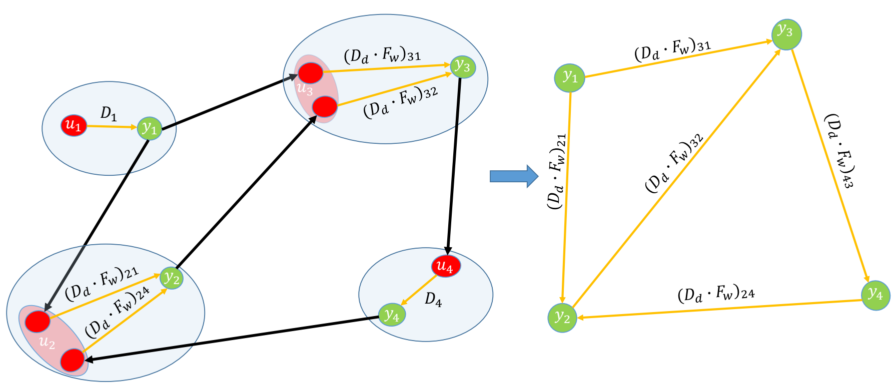

FIG. A.17. Applying the multiplication with F_w to the network from the Fig A.16

The block element $\left(D_d \cdot F_w\right)_{ij}^2$ defines the total impact of the output vector y_j onto the output vector y_i over the feedforward links of the system j and another system k that form a path between the internal nodes j and i (i.e. $\{k \in \mathscr{N} \mid w_{jk} = w_{ki} = 1\}$).

Finally, total impact of the output vector y_j onto the output vector y_i, over the feedforward links of the systems from the network (which is equivalent to the sum of all paths from the condensed output node y_j to the condensed output node y_i, over only the feedforward links, using the condensed graph representation of the network dynamics) is determined as follows:

$$\left(\sum_{l=1}^{\infty}\left(D_d \cdot F_w\right)^l\right)_{ij}$$

Under the assumption that all eigenvalues of the matrix $(D_d \cdot F_w)$ have absolute value smaller than 1, the total impact of the vector y_j onto the vector y_i, over the feedforward links of the systems from the network is defined by:

$$\left(I - D_d \cdot F_w\right)^{-1}_{ij}$$

In case there are no closed paths in G, the relation can be further simplified:

$$\left(\sum_{l=1}^{h}\left(D_d \cdot F_w\right)^l\right)_{ij}$$

where h is the hop-count of the longest walk in G. Finally, we conclude that the state matrix A_e of the network dynamics is explicitly determined as follows:

$$A_e = (B_d \cdot F_w) \cdot \sum_{i=0}^{\infty} (D_d \cdot F_w)^i \cdot C_d + A_d$$

where $i = 0$ regards the case when the two nodes/systems are direct neighbours. In case there is no direct relation between the input vector u_i and the output vector y_i of each system in the network ($i \in \mathscr{N}$), i.e. $D_d = O$, the state matrix A_e is defined as follows:

$$A_e = (B_d \cdot F_w) \cdot C_d + A_d$$

F.3 *Matrix B_e*

The input matrix B_e of the network dynamics has dimensions $\sum_{i=1}^{N} n_i \times M$ and is composed of $N \times r$ block elements. The $n_i \times \mu_j$ block element $\left(B_e\right)_{ij}$ determines the relation between the state vector x_i and the external input vector η_j. Following the condensed graph representation of the network dynamics, we conclude that the matrix $\left(B_e\right)^T$ defines the weights of the input links, which impose time delay while transmitting the signal.

The block element $\left(B_e\right)_{ij}$ is determined as a sum of all the paths from the condensed input node j to the condensed state node x_i of the system i in G_e, under the constraint that each path imposes fixed time delay equal to the one discrete-time instant. Furthermore, each such a path ends with the condensed input links of the system i (input links impose time delay), thus the remaining part of each path transmits the signal instantaneously. We recognize here two scenarios.

The first scenario is when the internal node i receives a direct link from the j-th input node of the extended network (i.e. $\phi_{ji} = 1$). As an example, the internal node 1 of the network presented in Fig A.18 receives a direct link from the input node 1 of the network. Thus, the block-element $\left(B_e\right)_{11}$ contains the μ_1 columns of the input matrix B_1, that are related to the external input vector η_1.

Therefore, we introduce the $(S_w + S_\phi) \times M$ matrix F_ϕ, consisting of $(L_w + L_\phi) \times r$ block elements, that are defined as follows:

$$\left(F_\phi\right)_{ij} = \begin{cases} I_{(s_w+s_\phi)_i} & \text{if } \left(\Gamma_\phi\right)_{ij} = 1 \\ O_{(s_w+s_\phi)_i \times \mu_j} & \text{otherwise} \end{cases} \tag{A.2}$$

where each block row of the matrix F_ϕ is related to a certain link of the network, while each block column is related to a certain input node of the network. Analogously as for the matrix F_w, we combine the block rows into N new block rows, where the new i-th block row regards the links that are connected to the system i and is composed of m_i rows. In case the input node j is directly connected to the state

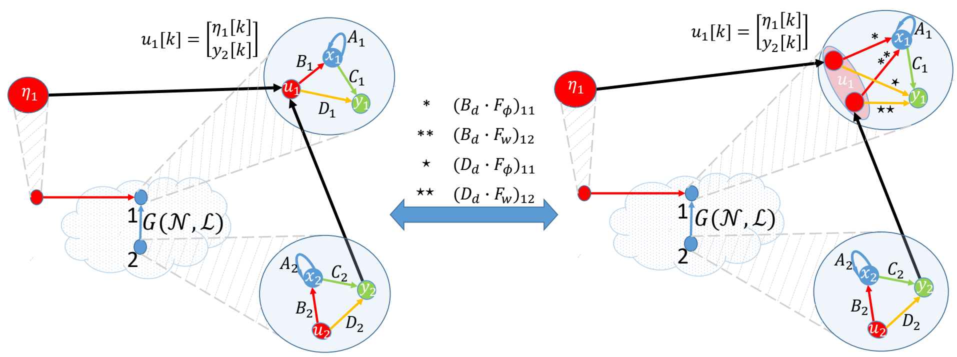

FIG. A.18. A condensed graph representation of the network dynamics, with $N = 2$ and $r = 1$

node i in G_e (i.e. $\phi_{ji} = 1$), the block element $\left(B_d \cdot F_\phi\right)_{ij} = B_i \cdot \left(F_\phi\right)_{ij}$ contains the μ_j columns of the matrix B_i that are related to the vector η_j, otherwise (i.e. $\phi_{ji} = 0$) it is a zero matrix. The analogous explanation holds for the block element $\left(D_d \cdot F_\phi\right)_{ij} = D_i \cdot \left(F_\phi\right)_{ij}$.

For the network from Fig A.18, we conclude:

$$B_e = \begin{bmatrix} \left(B_1 \cdot \left(F_\phi\right)_{11}\right)_{n_1 \times \mu_1} \\ O_{n_2 \times \mu_1} \end{bmatrix}$$

The second scenario is when there is a path from the j-th input node to the i-th internal node, over other internal nodes in G_e. Each such a path consists only of the condensed feedforward links of the nodes/systems that form a path from the input node j to the internal node i.

Finally, we conclude that the input matrix B_e of the network dynamics is explicitly determined as follows:

$$B_e = \left(B_d \cdot F_w\right) \cdot \sum_{i=0}^{\infty} \left(D_d \cdot F_w\right)^i \cdot \left(D_d \cdot F_\phi\right) + B_d \cdot F_\phi$$

where the first and the second term regards the second and the first scenario, respectively. In case the feedforward matrix of each system in the network is a zero matrix (i.e. $D_d = O$), the previous relation is significantly simplified:

$$B_e = B_d \cdot F_\phi$$

F.4 *Matrix C_e*

The output matrix C_e of the network dynamics has dimensions $P \times \sum_{i=1}^{N} n_i$ and is composed of $q \times N$ block elements. The $\rho_i \times n_j$ block element $\left(C_e\right)_{ij}$ defines the impact of the state vector x_j onto the external output vector ξ_i. The transposed output matrix $\left(C_e\right)^T$ defines the weights of the output links in the graph representation of the network dynamics.

Following the condensed graph representation of the network dynamics, we conclude that the $\rho_i \times n_j$ block element $\left(C_e\right)_{ij}$ is determined as a sum of all the paths from the condensed state node x_j to the condensed external output node ξ_i, under the constraint that each path introduces no time delay. Furthermore, each path starts with the condensed output links of the system j, while the remaining part of each path is composed of the condensed feedforward links of the systems that form a path from the internal node j to the output node i in G_e. We recognize here two scenarios.

The first scenario is when the internal node j is directly connected to the output node i of G_e (i.e. $\psi_{ji} = 1$). In this case, the impact of the state vector x_j onto the external output vector ξ_i is defined by C_j. As an example, there is a direct link from the internal node 1 to the output node 1 in the network presented in Fig A.19.

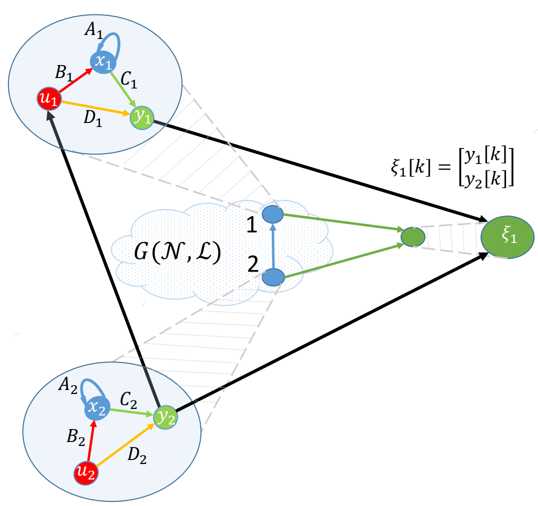

FIG. A.19. A condensed graph representation of the network dynamics, where $N = 2$ and $q = 1$

The second scenario is when there is a path from the internal node j to the output node i, over other internal nodes in G_e. These paths start with the condensed output links of the system j and continue with the condensed feedforward links of the systems that form a path in G_e between the internal node j and the output node i.

We introduce the $(S_\psi + S_z) \times P$ matrix F_ψ, composed of the $(L_\psi + L_z) \times N$ block elements, that are defined as follows:

$$\left(F_{\psi}\right)_{ij} = \begin{cases} I_{(s_{\psi}+s_z)_i} & \text{if } \left(\Gamma_{\psi}\right)_{ij} = 1 \\ O_{(s_{\psi}+s_z)_i \times p_j} & \text{otherwise} \end{cases} \tag{A.3}$$

Each block row of the F_{ψ} is related to a certain output or external link of the network. First $\left(l_{\psi}\right)_1 + \left(l_z\right)_1$ block rows are related to the links that are connected to the output node 1, next $\left(l_{\psi}\right)_2 + \left(l_z\right)_2$ block rows are related to the links that are connected to the output node 2 and so on. The last $\left(l_{\psi}\right)_q + \left(l_z\right)_q$ block rows are related to the links that finish in output node q. By combining those block rows related to the links connected to a certain output node, we can say that the matrix F_{ψ} is composed of the $q \times N$ block elements, where the i-th block row regards the links connected to the i-th output node of the network and contains ρ_i rows.

The output node i of G_e receives $\left(l_{\psi}\right)_i$ output links, as well as $\left(l_z\right)_i$ external links. Thus, the vector ξ_i is composed of all the external input vectors and the output vectors of the systems that are directly connected to the output node i. In case there is a direct link from the internal node j to the i-th output node (i.e. $\psi_{ji} = 1$), we have $\left(F_{\psi} \cdot C_d\right)_{ij} = \left(F_{\psi}\right)_{ij} \cdot C_j = C_j$, otherwise (i.e. $\psi_{ji} = 1$) we the block element $\left(F_{\psi} \cdot C_d\right)_{ij}$ is a zero matrix. The analogous explanation holds for the block element $\left(F_{\psi} \cdot D_d\right)_{ij} = \left(F_{\psi}\right)_{ij} \cdot D_j$.

For the network from the Fig A.19 we conclude:

$$\left(C_e\right)_{11} = \begin{bmatrix} \left(C_1\right)_{p_1 \times n_1} \\ O_{p_2 \times n_1} \end{bmatrix}, \left(C_e\right)_{12} = \begin{bmatrix} \left(D_1 \cdot C_2\right)_{p_1 \times n_2} \\ \left(C_2\right)_{p_2 \times n_2} \end{bmatrix}$$

Finally, we conclude that the output matrix C_e of the network dynamics is explicitly determined as follows:

$$C_e = F_{\psi} \cdot \sum_{i=0}^{\infty} (D_d \cdot F_w)^i \cdot C_d$$

where $i = 0$ regards the first scenario. In case there is no direct relation between the input vector u_i and the output vector y_i of each system in the network ($i \in \mathcal{N}$), i.e. $D_d = O$, the output matrix C_e is defined as follows:

$$C_e = F_{\psi} \cdot C_d$$

F.5 *Matrix D_e*

The feedforward matrix D_e of the network dynamics has dimensions $P \times M$ and is composed of the $q \times r$ block elements. The $\rho_i \times \mu_j$ block element $\left(D_e\right)_{ij}$ defines direct relation between the j-th external input vector η_j and the i-th external output vector ξ_i. The transposed feedforward matrix $\left(D_e\right)^T$ defines the weights of the feedforward links (feedforward links impose no time delay) in the graph representation of the network dynamics.

Using the condensed graph representation of the network dynamics, we determine the block element $\left(D_e\right)_{ij}$ as a sum of all the paths from the j-th condensed external input node η_j to the i-th condensed external output node ξ_i, under the constraint that each path imposes no time delay while transmitting the signal. We recognize here two scenarios.

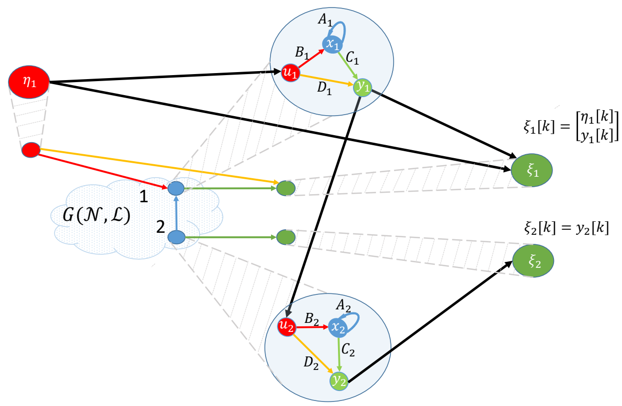

FIG. A.20. A condensed graph representation of the network dynamics, with $N = 2$, $r = 1$ and $q = 2$

The first scenario is when there is a direct link from the j-th input node to the i-th output node in G_e (i.e. $z_{ji} = 1$). An example is the external link from the input node 1 to the output node 1 of the network presented in Fig A.20. Therefore, we introduce the $(S_\psi + S_z) \times M$ matrix F_z, composed of the $(L_\psi + L_z) \times r$ block elements, defined as follows:

$$\left(F_z\right)_{ij} = \begin{cases} I_{(s_\psi+s_z)_i} & \text{if } \left(\Gamma_z\right)_{ij} = 1 \\ O_{(s_\psi+s_z)_i \times \mu_j} & \text{otherwise} \end{cases} \tag{A.4}$$

Analogously as for the matrix F_ψ, we combine block rows of the matrix F_z into new q block rows, where the i-th new block row regards the links connected to the i-th output node of G_e and contains ρ_i rows. The $\rho_i \times \mu_j$ block element $\left(F_z\right)_{ij}$ is identity matrix iff there is a direct link from the j-th input node to the i-th output node of G_e (i.e. $z_{ji} = 1$), otherwise (i.e. when $z_{ji} = 0$) the block element $\left(F_z\right)_{ij}$ is a zero matrix.

The second scenario is when there is a path from the j-th input node to the i-th output node of, over the internal nodes of the extended network G_e. These paths consist of the condensed feedforward links of the systems that form a path from the j-th input node to the i-th output node.

For the network from Fig A.20, we conclude:

$$\left(D_e\right)_{11} = \begin{bmatrix} I_{\mu_1 \times \mu_1} \\ \left(D_1\right)_{p_1 \times \mu_1} \end{bmatrix}, \left(D_e\right)_{21} = \begin{bmatrix} \left(D_2 \cdot D_1\right)_{p_2 \times \mu_1} \end{bmatrix}$$

Finally, we conclude that the feedforward matrix D_e of the network dynamics is determined explicitly as follows:

$$D_e = F_\psi \cdot \sum_{i=0}^{\infty} \left(D_d \cdot F_w\right)^i \cdot \left(D_d \cdot F_\phi\right) + F_z$$

where the first and the second term regard the second and the first scenario, respectively. In case there is no direct relation between the input vector u_i and the output vector y_i for each system of the network (i.e. $D_d = O$), the previous relation is significantly simplified:

$$D_e = F_z$$

which completes the intuitive explanation of the Theorem 1.

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