

Node-Reliability: Monte Carlo, Laplace, and Stochastic Approximations and a Greedy Link-Augmentation Strategy

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Abstract—The node-reliability polynomial $nRel_G(p)$ measures the probability that a connected network remains connected given that each node functions independently with probability p . Computing node-reliability polynomials $nRel_G(p)$ exactly is NP-hard. Here we propose efficient approximations. First, we develop an accurate Monte Carlo simulation, which is accelerated by incorporating a Laplace approximation that captures the polynomial's main behavior. We also introduce three degree-based stochastic approximations (Laplace, arithmetic, and geometric), which leverage the degree distribution to estimate $nRel_G(p)$ with low complexity. Beyond approximations, our framework addresses the reliability-based Global Robustness Improvement Problem (k -GRIP) by selecting exactly k links to add to a given graph so as to maximize its node reliability. A Greedy Lowest-Degree Pairing Link Addition (Greedy-LD) Algorithm, is proposed which offers a computationally efficient and practically effective heuristic, particularly suitable for large-scale networks.

Index Terms—network robustness, node failure, probabilistic graph, reliability polynomial

I. INTRODUCTION

RELIABILITY research in network science is concerned with the estimation of the probability that the residual network remains operational after the failure of some components [1]. In 1956, Moore and Shannon [2] proposed a probabilistic model for network reliability. Based on the types of components that can fail, network reliability can be classified into two categories:

Network reliability w.r.t. link failures: defined as the probability that the nodes of a connected graph G remain connected if each link is independently operational with probability p , assuming the nodes of the graph are perfectly reliable [3]. This type of network reliability can be expressed as a so-called reliability polynomial:

$$Rel_G(p) = \sum_{j=0}^L F_j(G) (1-p)^j p^{L-j}, \quad (1)$$

where $F_j(G)$ is the number of sets of j links whose removal leaves G connected, and $F_0(G) = 1$.

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Network reliability w.r.t. node failures: defined as the probability that the operational nodes of a connected graph G remain connected if each node is independently operational with probability p , assuming that the links of the graph are perfectly reliable [3]. This type of network reliability can be expressed as the node reliability polynomial:

$$nRel_G(p) = \sum_{k=0}^N S_k(G) p^k (1-p)^{N-k}, \quad (2)$$

where $S_k(G)$ denotes the number of induced connected sub-graphs with k nodes.

Most studies on network reliability focus on link failures. This paper investigates node failures, assuming that each node remains operational independently with a uniform probability p . This assumption is justified in scenarios where nodes share similar physical characteristics, such as battery depletion in wireless sensor networks, random hardware failures in data centers, or telecommunication switching systems [4], [5]. The uniform probability provides a baseline scenario for evaluating network robustness, allowing straightforward comparisons across network topologies. Although the real-world node failure probabilities may vary, the uniform assumption simplifies initial robustness analysis and serves as a foundation for extensions to heterogeneous scenarios. Additionally, the assumption that links are perfectly reliable is appropriate when node robustness dominates reliability concerns, and link failures are comparatively negligible. Examples include wireless sensor networks, where link existence depends solely on node status, or data center networks, where node failures significantly outweigh link failures.

Compared to robustness metrics such as connectivity, algebraic connectivity, [6] natural connectivity, [7] network criticality, [8] path diversity, and spectral gap, [9] the node reliability polynomial provides distinct advantages. First, it explicitly incorporates the probabilistic nature of random node failures, intuitively interpreting network robustness as the probability of remaining interconnected under any node failure scenario. Traditional connectivity metrics, like algebraic connectivity and spectral gap, focus primarily on worst-case partitioning, lacking sensitivity to incremental probabilistic changes. Second, unlike natural connectivity and network criticality, node reliability quantitatively captures the cumulative effect of all possible failure scenarios, aiding precise evaluation and optimization of redundancy. Lastly, while path diversity assesses redundancy between specific node pairs,

node reliability evaluates global connectivity involving all active nodes, thus offering a more comprehensive robustness assessment for practical communication networks and critical infrastructures.

The problems of computing the reliability polynomial $\text{Rel}_G(p)$ and node reliability polynomial $\text{nRel}_G(p)$ are NP-hard [10]–[12]. Closed-form analytic expressions for the node reliability polynomial only exist for some specific graph topologies [13]. Appendix D and Table I contains some examples. Various Monte Carlo methods give accurate estimations for the node reliability polynomial, but suffer from a high computational complexity [14]–[16]. For small- to medium-sized graphs, exact evaluation can also be performed with Binary Decision Diagram (BDD) techniques, such as the efficient vertex-reliability construction proposed by Kawahara *et al.* [17]. The reliability polynomial captures crucial information about a network's connectivity by encoding all possible cut sets—the sets of links or nodes whose removal would disconnect the network. The reliability polynomial thus serves as a comprehensive measure of a network's global robustness. Networks with higher values of the reliability polynomial, under the same operational probability p , tend to be more resilient to disconnection, allowing comparisons between different network topologies. In addition to a structural analysis, the reliability polynomial plays a key role in network design [1], [18], [19]. The reliability polynomial can identify critical nodes or links whose addition or removal significantly affects the overall reliability [19]. For instance, adding links can enhance reliability in communication networks by increasing redundancy, whereas removing specific links can effectively contain the spread of diseases [19]. The reliability polynomial finds practical applications in fields such as communication networks, infrastructure systems, and public health [19]–[21].

This paper first introduces a Laplace approximation for the node reliability polynomial in Section II. In Section III, we propose a Monte Carlo method for approximating node reliability polynomials. The proposed Monte Carlo method is inspired by a recent fast approach designed for network reliability polynomials [22]. Additionally, the Monte Carlo method is combined with the Laplace approximation to develop a new hybrid approach, referred to as the Laplace Monte Carlo method. Section IV introduces three degree-based stochastic approximations (Laplace, arithmetic and geometric) for the node-reliability polynomial. The relation between the reliability polynomial $\text{Rel}_G(p)$ and the node reliability polynomial $\text{nRel}_G(p)$ is also analyzed. Section V discusses a practical application of network reliability to link-augmentation (k -GRIP) optimisation. Notation and all variables are summarized in Appendix A. Additional analytical results for Erdős–Rényi and Random Geometric Graphs are provided in Appendix E.

II. THE LAPLACE APPROXIMATION FOR THE NODE RELIABILITY POLYNOMIAL $\text{nRel}_G(p)$

The number of combinations of k different nodes out of N nodes is the binomial coefficient $\binom{N}{k} = \frac{N!}{k!(N-k)!}$. We define the number $C_j(G)$ as the number of subsets of j nodes whose removal disconnects the graph G . Every subset of k nodes in

G must either be connected or disconnected, leading to the following relationship:

$$S_k(G) + C_{N-k}(G) = \binom{N}{k} \quad (3)$$

After substituting $S_k(G) = \binom{N}{k} - C_{N-k}(G)$ into Eq. (2), and applying Newton's binomial theorem, $(a + b)^N = \sum_{k=0}^N \binom{N}{k} a^{N-k} b^k$, we obtain:

$$\text{nRel}_G(p) = 1 - \sum_{k=1}^N C_{N-k}(G) p^k (1-p)^{N-k} \quad (4)$$

Hence, the (all-terminal) node reliability polynomial can be expressed both in the “ S -form” and in the “ C -form” as

$$\begin{aligned} \text{nRel}_G(p) &= \sum_{k=0}^N S_k(G) p^k (1-p)^{N-k} \\ &= 1 - \sum_{j=1}^N C_j(G) p^{N-j} (1-p)^j \end{aligned} \quad (5)$$

where $S_k(G)$ counts the number of induced connected subgraphs on k nodes and $C_j(G)$ counts the number of vertex cut sets with j nodes, which is the number of subsets of j nodes whose removal disconnect the graph. We can express the node reliability polynomials in the “ S -form” and “ C -form” in *binomial forms* as well:

$$\begin{aligned} \text{nRel}_G(p) &= \sum_{k=0}^N \binom{N}{k} s_k(G) p^k (1-p)^{N-k} \\ &= 1 - \sum_{j=0}^N \binom{N}{j} c_j(G) p^{N-j} (1-p)^j \end{aligned} \quad (6)$$

where $s_k(G) = \frac{S_k(G)}{\binom{N}{k}}$ and $c_j(G) = \frac{C_j(G)}{\binom{N}{j}}$ are the fractions of induced connected subgraphs with k nodes and vertex cut sets of graph G with j nodes amongst all possible node combinations of k nodes and j nodes from N nodes. In other words, $s_k(G)$ equals the probability that the residual network remains connected after removing $N-k$ nodes and $c_j(G)$ is the probability that the residual part of graph G is disconnected after removing j nodes.

Although computing $S_k(G)$ and $C_j(G)$ is NP-hard, the node reliability polynomials $\text{nRel}_G(p)$ can still be approximated by estimating the probabilities $s_k(G)$ and $c_j(G)$. In this paper, we propose a Laplace Monte-Carlo approximation of $\text{nRel}_G(p)$ based on the “ C -form” of the node reliability polynomial and a stochastic approximation of $\text{nRel}_G(p)$ based on the “ S -form” of the node reliability polynomial.

The term $\binom{N}{k} p^k (1-p)^{N-k}$ represents the probability density function (pdf) of the binomial distribution. The Central Limit theorem states [23], [24] that the binomial distribution approaches the Gaussian distribution for large N . For large

N , the “ S -” and “ C -form” of the node reliability polynomials can then be approximated as:

$$\begin{aligned} \text{nRel}_G(p) &= \sum_{k=0}^N \binom{N}{k} s_k(G) p^k (1-p)^{N-k} \\ &\simeq \int_0^N s_k(G) \frac{\exp\left(-\frac{(Np-k)^2}{2Np(1-p)}\right)}{\sqrt{2\pi Np(1-p)}} dk \end{aligned} \quad (7)$$

Substituting $x = \frac{k}{N}$ transforms the integral in (7) into

$$\int_0^1 s_{Nx}(G) \frac{1}{\sqrt{2\pi} \sqrt{\frac{p(1-p)}{N}}} \exp\left(-\frac{(p-x)^2}{2\frac{p(1-p)}{N}}\right) dx \quad (8)$$

If we define $\tilde{\mu} = p$, and $\tilde{\sigma} = \sqrt{\frac{p(1-p)}{N}}$, then:

$$\text{nRel}_G(p) \simeq \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \int_0^1 s_{Nx}(G) \exp\left(-\frac{(\tilde{\mu}-x)^2}{2\tilde{\sigma}^2}\right) dx \quad (9)$$

The Gaussian pdf $\frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\left(-\frac{(\tilde{\mu}-x)^2}{2\tilde{\sigma}^2}\right)$ serves as an approximation of the Dirac delta function as [9, Sec. 7.1]:

$$\delta(x - \tilde{\mu}) = \lim_{\tilde{\sigma} \rightarrow 0} \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \exp\left(-\frac{(\tilde{\mu}-x)^2}{2\tilde{\sigma}^2}\right) \quad (10)$$

Assuming that $\tilde{\sigma}$ tends to 0, the node reliability polynomial can be approximated by the following expression:

$$\begin{aligned} \text{nRel}_G(p) &\simeq \frac{1}{\sqrt{2\pi}\tilde{\sigma}} \int_0^1 s_{Nx}(G) \exp\left(-\frac{(\tilde{\mu}-x)^2}{2\tilde{\sigma}^2}\right) dx \\ &\simeq \int_0^1 s_{Nx}(G) \delta(x - \tilde{\mu}) dx = s_{N\tilde{\mu}}(G) \end{aligned} \quad (11)$$

where $\tilde{\mu} = p$.

In summary, we call the approximation

$$\text{nRel}_G(p) \simeq \text{nRel}_G^{\text{Lap}}(p) = s_{Np}(G) \quad (12)$$

the Laplace approximation of the node reliability polynomial. Following a similar derivation, the Laplace approximation of the “ C -form” of the node reliability polynomial is given by

$$\text{nRel}_G^{\text{Lap}}(p) = 1 - c_{N(1-p)}(G) \quad (13)$$

Based on the analysis in Appendix B, the total error of the Laplace approximation is $o\left(\frac{1}{\sqrt{N}} + \frac{s''_{Np}(G)}{N}\right)$, where $s''_{Np}(G)$ is the second derivative of $s_{Np}(G)$ about p . When $s_{Np}(G)$ varies more gently (i.e. $s''_{Np}(G)$ is of smaller order than $o(N)$), the second term decreases, and the approximation becomes more accurate. Meanwhile, for a fixed p , the error decreases as N grows, further improving the accuracy. Hence, for sufficiently large N , the Laplace approximation in Eq. (12) yields increasingly accurate results.

III. MONTE CARLO METHOD FOR APPROXIMATING THE NODE RELIABILITY POLYNOMIAL $\text{nRel}_G(p)$

The Monte Carlo method for estimating node reliability polynomials is based on a node deletion process, where at each time step a randomly selected node is removed. In a given graph, nodes are removed one by one until all nodes are eliminated. After each removal, the residual network is checked to determine whether it remains connected. By repeating the node deletion process M times, the number of cases R_j in which the removal of j nodes disconnects the residual graph is obtained for each node $j \in [1, N]$. When the number of realizations M is large, the probability that the removal of j nodes disconnects the graph is approximately $\tilde{c}_j(G) = \frac{R_j}{M}$. Thus,

$$c_j(G) = \frac{C_j(G)}{\binom{N}{j}} \simeq \tilde{c}_j(G) \quad (14)$$

The C -form (4) of the node reliability polynomial $\text{nRel}_G(p) = 1 - \sum_{j=1}^N \binom{N}{j} c_j(G) p^{N-j} (1-p)^j$ can be approximated by:

$$\text{nRel}_{G,\text{MC}}(p) \simeq 1 - \sum_{j=1}^N \binom{N}{j} \tilde{c}_j(G) p^{N-j} (1-p)^j \quad (15)$$

where we denote the Monte Carlo approximation of $\text{nRel}_G(p)$ by $\text{nRel}_{G,\text{MC}}(p)$. The computational complexity of the Monte Carlo method is $\mathcal{O}(MN)$. Based on the analysis in Appendix C, the mean squared error (MSE) of the Monte Carlo approximation $\text{nRel}_{G,\text{MC}}(p)$ is $\text{MSE}[\text{nRel}_{G,\text{MC}}(p)] = o\left(\frac{1}{M}\right)$. Fig. 1 demonstrates the inverse power-law relationship between MSE and number of simulations M by showing a linear decay trend in the log-log plot of M versus MSE, consistent with a convergence rate close to $\mathcal{O}(1/M)$. The proba-

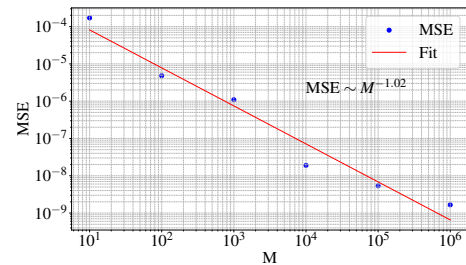


Fig. 1: MSE of the Monte Carlo approximation $\text{nRel}_{G,\text{MC}}(p)$ as a function of number of simulations M for complete graph K_9 with a pendant node K_{10}^* . The slope of the red fit curve is -1.02

bilistic error bound is $\Pr(|\text{nRel}_{G,\text{MC}}(p) - \text{nRel}_G(p)| > \epsilon) \leq 2 \exp\left(-CM\epsilon^2 \sqrt{Np(1-p)}\right)$, where C is a positive constant. The error probability decays exponentially in M , ϵ^2 , and $\sqrt{Np(1-p)}$. For a network with a number of nodes N and a prescribed error threshold ϵ , the probability that the Monte Carlo estimate $\text{nRel}_{G,\text{MC}}(p)$ deviates from the true value $\text{nRel}_G(p)$ by more than ϵ decreases exponentially with a fixed number M of simulations. For a fixed error threshold ϵ , a larger network size N leads to a smaller required number of simulations M to achieve the same error probability.

TABLE I: Network reliability polynomials for some simple networks

Network (on N nodes)	$\text{nRel}_G(p)$
Complete graph K_N	$\text{nRel}_G(p) = 1 - (1-p)^N$
Complete graph K_{N-1} with a pendant node K_N^*	$\text{nRel}_G(p) = p^2 + p(1-p)^{N-1} + (1-p)(1-(1-p)^{N-1})$
Cycle graph C_N	$\text{nRel}_G(p) = \frac{Np(p^{N-1} - (1-p)^N)}{2p-1} - (N-1)p^N \left(\text{nRel}_G(0.5) = \frac{N^2 - N + 1}{2^N} \right)$
Path graphs P_N	$\text{nRel}_G(p) = \frac{Np(1-p)^{N+1} - (N+1)p^2(1-p)^N + p^{N+2}}{(1-2p)^2} \left(\text{nRel}_G(0.5) = \frac{N(N+1)}{2^{N+1}} \right)$
Star graph S_N	$\text{nRel}_G(p) = p + (N-1)p(1-p)^{N-1}$
Star graph S_{N-1} with a pendant node S_N^*	$\text{nRel}_G(p) = p^3 + p^2(1-p)^{N-2} + p(1-p)^{N-1} + (1-p)(p + (N-2)p(1-p)^{N-2})$

The Monte Carlo method is applied to several simple graphs, for which explicit closed-form analytical expressions of node reliability polynomials are known, see Table I and Appendix D. The graphs are the complete graph K_N , the complete graph K_{N-1} with a pendant node, denoted by K_N^* (N nodes), the cycle graph C_N , the path graph P_N , the star graph S_N (N nodes) and the star graph S_{N-1} with a pendant node, denoted by S_N^* (N nodes). The node reliability polynomials and the result of the Monte Carlo simulations are depicted in Fig.2 and Table III, which demonstrate that the Monte Carlo approximation is accurate for the considered graphs. In the remainder of this paper, the Monte Carlo method is used as a benchmark to evaluate other approximations of the node reliability polynomial.

The analysis in Section II has shown that, if the number of nodes N in the graph G is large, the node reliability polynomial can be approximated by Eq. (13) $\text{nRel}_G(p) \simeq 1 - c_N(1-p)$. The combination of the Monte Carlo method and the Laplace approximation is referred to as the Laplace Monte Carlo method, denoted as $\text{nRel}_{G,MC}^{\text{Lap}}(p) = 1 - c_N(1-p)$. The error of the Laplace Monte Carlo method combines the errors from the Laplace approximation and Monte Carlo sampling, and is therefore of order $o\left(\frac{1}{\sqrt{N}} + \frac{s''_{Np}(G)}{N} + \frac{1}{\sqrt{M}}\right)$. Fig.3 presents a comparison between the theoretical values of the node reliability polynomial $\text{nRel}_G(p)$ and the Laplace Monte Carlo approximation, represented as $1 - c_N(1-p)$. Fig. 3 indicates that the Laplace Monte Carlo approximation fits the theoretical values more closely as the size of the graph increases. Table III shows the MSE, mean absolute error (MAE), and maximum error of the Laplace Monte Carlo approximation $\text{nRel}_{G,MC}^{\text{Lap}}(p)$. Comparing Figs. 3a, 3b, and 3d, we observe that, with the number of simulations M held fixed, both the MSE and MAE decline as the graph size N increases. Conversely, a comparison between Figs. 3b and 3c shows that, for a fixed graph size N , increasing M likewise reduces both MSE and MAE. The maximum error is contributed by the difference of $\text{nRel}_G(p)$ and $1 - c_{Np}$ at $p = \frac{1}{N}$, the larger the number of nodes N , the larger the second derivative value $s''_{Np}(G)$ at $p = \frac{1}{N}$, which contributes to a larger maximum error.

IV. STOCHASTIC APPROXIMATION FOR THE NODE RELIABILITY POLYNOMIAL

Our previous work [25] has introduced a stochastic approximation for the reliability polynomial $\text{Rel}_G(p)$. In this paper, we extend that approach towards a stochastic approximation for the node reliability polynomial $\text{nRel}_G(p)$.

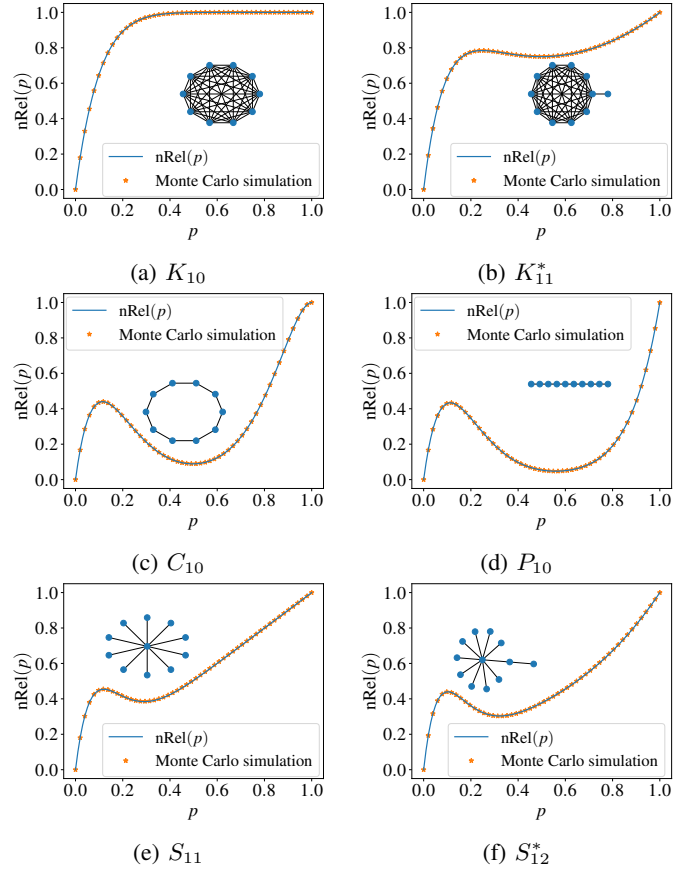


Fig. 2: Monte Carlo simulations ($M = 10000$) and exact values of node reliability polynomials for different graphs. The corresponding error metrics are summarized in Table III.

We denote the random residual graph with the failure of $N - k$ randomly selected nodes in G as $\hat{G}_k$. The probability that the residual graph $\hat{G}_k$ is still connected equals $\Pr[\hat{G}_k \text{ is connected}] = s_k(G)$.

For a given graph, the implication $\{G \text{ is connected}\} \Rightarrow \{D_{\min} \geq 1\}$, where the minimum degree is $D_{\min} = \min_{\text{all nodes } \in G} D$, is always true. However, the opposite implication does not generally hold, because it is possible for a network to be composed of several disconnected clusters where each node has a minimum degree greater than 1. Van der Hofstad [26] proves that for an Erdős-Rényi(ER) graph with large N and certain link density p_l which depends on N , the opposite implication $\{D_{\min} \geq 1\} \Rightarrow \{G \text{ is connected}\}$ holds almost for sure. For other network models with large N

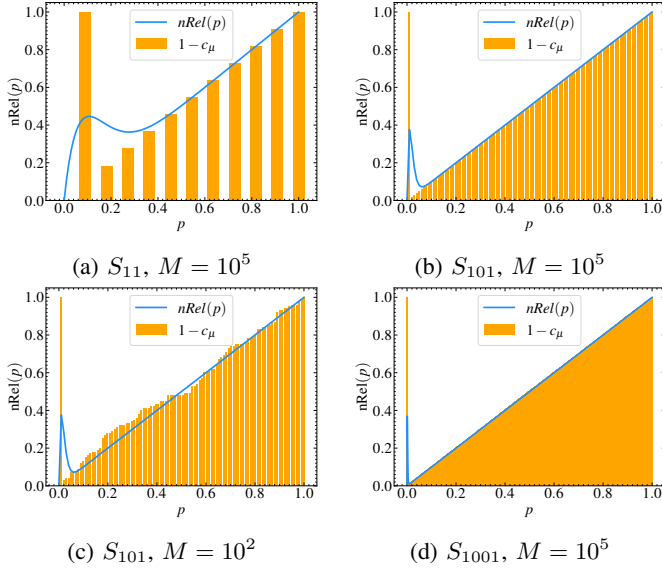


Fig. 3: The analytical expressions and Laplace Monte Carlo simulation results $1 - c_{N(1-p)}$ for the node reliability polynomial for star graphs S_{N-1} under different N and simulation sizes M . Error metrics are listed in Table III.

and high link density p_l , the equivalence $\{D_{\min} \geq 1\} \iff \{G \text{ is connected}\}$ also holds [27]–[29]. The main hypothesis of the stochastic approximation is that

$$\Pr[\hat{G}_k \text{ is connected}] = \Pr[\hat{D}_{\min} \geq 1] + o(1) \quad (16)$$

where $\hat{D}_{\min} = \min_{\text{all nodes } \in \hat{G}_k} \hat{D}$. Let $\Pr[D = k]$ be the probability that a randomly chosen node in the graph G has degree k . The probability generating function (pgf) of the node degree D in the graph G is defined [23] as:

$$\varphi_D(z) = E[z^D] = \sum_{j=0}^{N-1} \Pr[D = j] z^j \quad (17)$$

If the number of operational nodes is k , the probability that all neighbors of a node with degree j fail independently of each other equals $(1 - \frac{k}{N})^j$. Consequently, the probability that a randomly chosen residual node i in $\hat{G}$ is isolated $\Pr[d_i = 0]$ equals $\varphi_D(1 - \frac{k}{N})$ [25]. The probability that the minimum degree $D_{\min}$ is larger than 0 is approximated by

$$\Pr[\hat{D}_{\min} \geq 1] \simeq \prod_{i=1}^k (1 - \Pr[d_i = 0]) \simeq (1 - \Pr[d_i = 0])^k \quad (18)$$

The probability that the residual graph $\hat{G}_k$ remains connected is approximated as $\Pr[\hat{G}_k \text{ is connected}] \simeq \Pr[\hat{D}_{\min} \geq 1]$

$$\Pr[\hat{G}_k \text{ is connected}] \simeq \left(1 - \varphi_D\left(1 - \frac{k}{N}\right)\right)^k \quad (19)$$

The definition of the coefficients $S_k(G)$ in the “ S -form” in (6) of the node reliability polynomial indicates that

$$S_k(G) = \binom{N}{k} s_k(G) \simeq \binom{N}{k} \left(1 - \varphi_D\left(1 - \frac{k}{N}\right)\right)^k \quad (20)$$

Substituting (20) to the “ S -form” of node reliability polynomial Eq. (2), leads to the approximation of the “ S -form” node reliability polynomial $\text{nRel}_G(p)$ as

$$\text{nRel}_G(p) \simeq \sum_{n=0}^N \binom{N}{n} \left(1 - \varphi_D\left(1 - \frac{n}{N}\right)\right)^n p^n (1-p)^{N-n} \quad (21)$$

When N is large, the value of the node reliability polynomial $\text{nRel}_G(p)$ can be approximated by $s_{Np}(G)$ as:

$$\begin{aligned} \text{nRel}_G(p) &\simeq s_{Np}(G) = \Pr[\hat{G}_{Np} \text{ is connected}] \\ &\simeq (1 - \varphi_D(1-p))^{Np} \end{aligned} \quad (22)$$

Following the approach in our previous work [25], the stochastic approximation of the node reliability polynomial is denoted as:

$$\overline{\text{nRel}}_G^{\text{Lap}}(p) = (1 - \varphi_D(1-p))^{Np} \quad (23)$$

The equivalence $\{G \text{ is connected}\} \iff \{D_{\min} \geq 1\}$, where the minimum degree is $D_{\min} = \min_{\text{all nodes } \in G} D$ holds when the number of nodes N and the link density p_l is large, thus the accuracy of stochastic approximation $\overline{\text{nRel}}_G^{\text{Lap}}(p)$ increases with the increase of N and p_l .

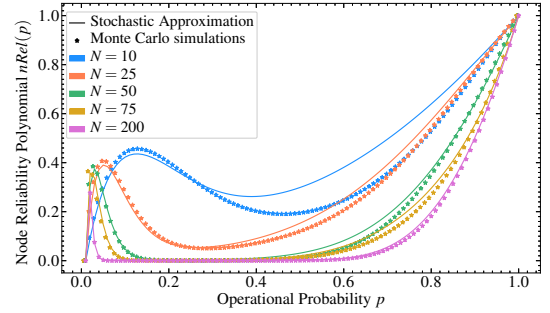


Fig. 4: Node reliability polynomial $\text{nRel}_G(p)$ obtained by the stochastic approximation and Monte Carlo simulations ($M = 10000$) for Erdős-Rényi graphs with different number of nodes N and link probability $p_l = \frac{\log N}{N}$ depending on the number of nodes.

We first perform the simulations on Erdős-Rényi graphs with different number of nodes N and link probability $p_l = \frac{\log N}{N}$. Fig. 4 demonstrates that the accuracy of the stochastic approximation increases with the size of the network. Derivations and analytical approximations of the node-reliability polynomial for Erdős-Rényi graphs are provided in Appendix E.

Fig. 5, 6, 7, and 8 depict the node reliability polynomial $\text{nRel}_G(p)$ obtained by stochastic approximation and Monte Carlo simulation for Barabási-Albert graphs, Erdős-Rényi graphs, 2D-lattice graphs and 3D-lattice graphs. We find that the accuracy of the stochastic approximation increases with link density p_l and the number of nodes N .

Fig. 10 shows the node reliability polynomial $\text{nRel}_G(p)$ obtained by stochastic approximation and Monte Carlo simulation for some real-world networks. The corresponding parameters for these networks are provided. We find that the stochastic approximation demonstrates high accuracy in

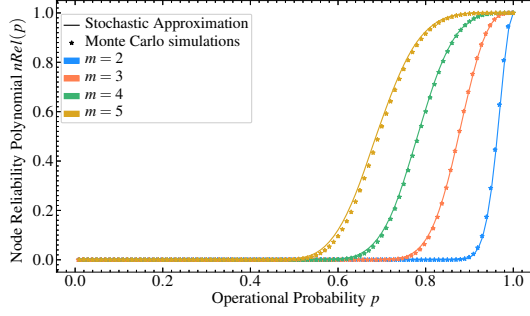


Fig. 5: Stochastic approximation and Monte Carlo simulations ($M = 10000$) of Barabási-Albert graphs with $N = 1000$ and different number of links added per step m

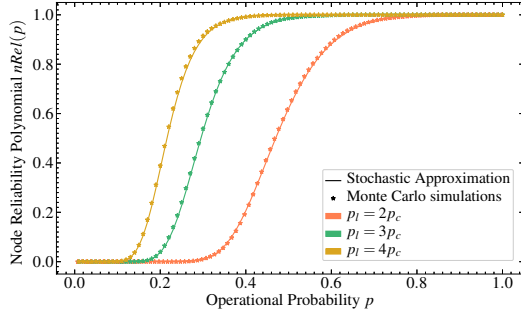


Fig. 6: Stochastic approximation and Monte Carlo simulations ($M = 10000$) for the Erdős-Rényi graphs with $N = 1000$ and critical link density $p_c \sim \frac{\log N}{N} = 0.0069$.

approximating the node reliability polynomial in real-world networks.

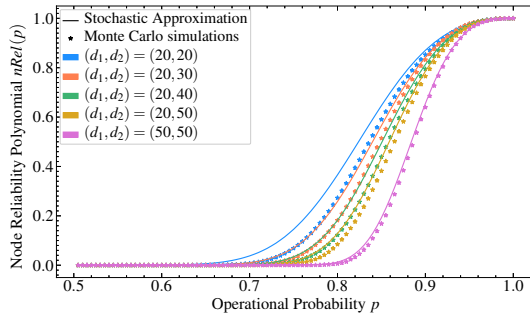


Fig. 7: Stochastic approximation and Monte Carlo simulations ($M = 10000$) for 2D-lattices with width d_1 and height d_2

We show in [25] that for graphs with a large number of nodes and high link density, the link reliability polynomial can be accurately approximated by:

$$\overline{\text{Rel}}_G(\tilde{p}) \simeq (1 - \varphi_D(1 - \tilde{p}))^N \quad (24)$$

where the variable $\tilde{p}$ denotes the probability of links being operational.

To evaluate the performance of our reliability approximation methods on structured communication topologies, we consider three representative network models: Torus [30], HyperX [31], and Fat-Tree [32]. These topologies are widely used in parallel

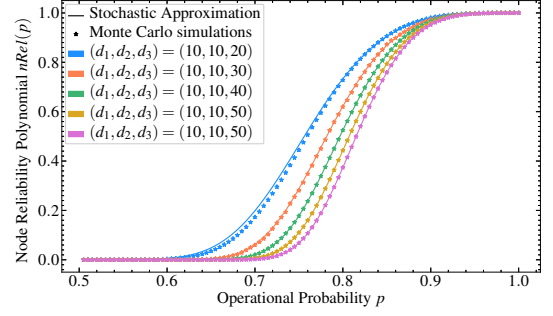


Fig. 8: The stochastic approximation and Monte Carlo simulations ($M = 10000$) for 3D-lattices with width d_1 , length d_2 and height d_3

computing and data center networks due to their scalability. Fig. 9 shows the node reliability polynomial $n\text{Rel}_G(p)$ obtained by stochastic approximation and Monte Carlo simulation. From the MSE and MAE value given in Table II, we find that the stochastic approximation shows a small error in computing the node reliability polynomial $n\text{Rel}_G(p)$ of the Torus network and HyperX network, but perform poorly on the Fat-Tree network. For a given node functional probability p , the node reliability $n\text{Rel}_G(p)$ of Fat-Tree network is always larger than the stochastic approximation $n\text{Rel}_G^{\text{Lap}}(p)$. The Fat-Tree topology provides high path redundancy, which makes it highly reliable against node failure.

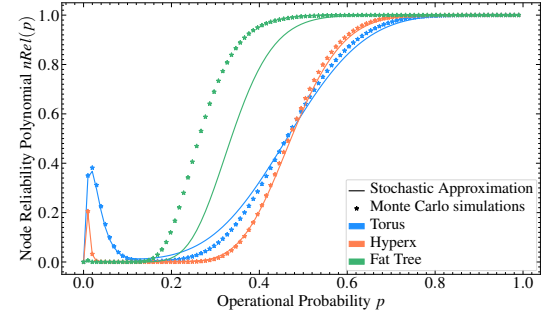


Fig. 9: Stochastic approximation and Monte Carlo simulations ($M = 10000$) for the node reliability of three structured network topologies: 3D Torus ($6 \times 6 \times 6$), HyperX (4D, 4 switches per dimension), and k -ary Fat-Tree ($k = 24$). The corresponding error metrics are reported in Table II.

TABLE II: Error metrics (MSE, MAE, $\Delta p_{99.999\%}$) for the structured topologies shown in Figure 9.

Network	MSE	MAE	$\Delta p_{99.999\%}$
Torus	4.835×10^{-4}	1.646×10^{-2}	0.042
HyperX	8.798×10^{-5}	5.392×10^{-3}	0.027
Fat-Tree	1.953×10^{-2}	7.052×10^{-2}	0.102

We define $p_{99.999\%}$ as the minimum node activation probability p such that the node reliability satisfies $n\text{Rel}_G(p) \geq 0.99999$. This value is critical for meeting high availability targets, such as the “five nines” standard in telecom systems.

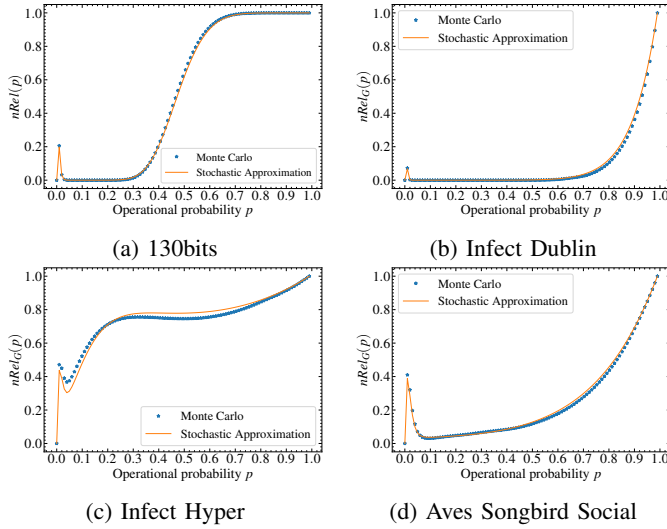


Fig. 10: Monte Carlo simulations and stochastic approximations for various real-world graphs. The corresponding error metrics are reported in Table III.

Our results show that the stochastic approximation closely estimates $p_{99.999\%}$, offering a fast and practical tool for guiding network design, redundancy planning, and equipment selection. We define $\Delta p_{99.999\%}$ as the absolute difference between the estimated values of $p_{99.999\%}$ obtained by the stochastic approximation and by Monte Carlo simulation. For the 3D Torus and HyperX, $\Delta p_{99.999\%}$ is 0.042 and 0.027, respectively, indicating good accuracy. For the Fat-Tree, the difference increases to 0.102, suggesting reduced precision on Fat-Tree topologies.

TABLE III: Error metrics (MSE, MAE, Maximum Error) for Figures 2, 3, and 10.

Subfigure	MSE	MAE	Max Error
2a	0	0	0
2b	1.182×10^{-6}	9.907×10^{-4}	1.753×10^{-3}
2c	2.373×10^{-6}	1.043×10^{-3}	3.288×10^{-3}
2d	8.512×10^{-7}	8.262×10^{-4}	1.733×10^{-3}
2e	2.485×10^{-5}	4.307×10^{-3}	7.247×10^{-3}
2f	3.146×10^{-6}	1.676×10^{-3}	2.149×10^{-3}
3a	3.228×10^{-2}	8.123×10^{-2}	0.559
3b	4.787×10^{-3}	1.188×10^{-2}	0.624
3c	5.817×10^{-3}	3.584×10^{-2}	0.624
3d	5.031×10^{-4}	2.785×10^{-3}	0.631
10a	1.328×10^{-3}	1.862×10^{-2}	2.215×10^{-2}
10b	1.712×10^{-4}	7.576×10^{-3}	9.617×10^{-3}
10c	7.827×10^{-4}	2.346×10^{-2}	3.151×10^{-2}
10d	1.803×10^{-4}	1.062×10^{-2}	1.291×10^{-2}

A. Arithmetic Stochastic Approximation $nRel_G^{arith}(p)$ and Geometric Stochastic Approximation $nRel_G^{geom}(p)$

The main hypothesis of stochastic approximation is $\Pr[\widehat{G}_k \text{ is connected}] = \Pr[\widehat{D}_{\min} \geq 1] + o(1)$. Sec. IV

gives a way to estimate the probability $\Pr[\widehat{D}_{\min} \geq 1]$ that the minimum degree in the residual network $\widehat{G}$ is larger than 0. In addition to the Laplace stochastic approximation $nRel_G^{Lap}(p)$, we provide two other stochastic approximations for the node reliability polynomials, the arithmetic stochastic approximation $nRel_G^{arith}(p)$ and geometric stochastic approximation $nRel_G^{geom}(p)$. Given a node i with degree d_i , we define the event $\{\text{Node } i \text{ fails}\} \cup \{\text{Node } i \text{ is active and at least one neighbor of } i \text{ is active}\}$ as X_i . Its probability is denoted by $f_i = 1 - p(1 - p)^{d_i}$. Under the independence assumption, the probability that every node is not isolated (i.e., each node experiences X_i) is $\Pr(\bigcap_{i=1}^N X_i) = \prod_{i=1}^N f_i$. However, when the events $\{X_i\}$ are not independent, we can correct the simple product by incorporating the dependencies via joint cumulants. In particular, according to the analysis in Appendix F, we have

$$\Pr\left[\bigcap_{i=1}^N X_i\right] = \prod_{i=1}^N f_i \times \left\{ 1 + \sum_{k=2}^N \frac{\kappa(X_{i_1}, X_{i_2}, \dots, X_{i_k})}{\prod_{m=1}^k f_{i_m}} \right\}. \quad (25)$$

where, $\kappa(X_{i_1}, X_{i_2}, \dots, X_{i_k})$ denotes the joint cumulant of the indicator variables of the events $X_{i_1}, X_{i_2}, \dots, X_{i_k}$, which quantifies the pure k th-order dependence among these events.

When the events $\{X_i\}$ are mutually independent, all joint cumulants for $k \geq 2$ vanish, and the expression in (25) reduces to the product $\prod_{i=1}^N f_i$. The resulting geometric stochastic approximation for the node reliability polynomial is

$$\overline{nRel}_G^{geom}(p) = \prod_{i=1}^N (1 - p(1 - p)^{d_i}) \quad (26)$$

When calculating the node reliability of large networks, concatenating the probabilities that each node is not isolated becomes computationally demanding. The arithmetic mean probability that a randomly selected node is not isolated node in $\widehat{G}$ is

$$P_{AM} = \frac{1}{N} \sum_{i=1}^N \Pr[X_i] \quad (27)$$

Substituting $\Pr[X_i] = f_i = 1 - p(1 - p)^{d_i}$ into Eq. (27), we obtain

$$P_{AM} = \frac{1}{N} \sum_{i=1}^N f_i = \frac{1}{N} \sum_{i=1}^N (1 - p(1 - p)^{d_i}) \quad (28)$$

which we rewrite as a sum over the nodal degrees by denoting n_j as the number of nodes with degree j and realizing that $\Pr[D = j] = \frac{n_j}{N}$,

$$\begin{aligned} P_{AM} &= \frac{1}{N} \sum_{j=0}^{N-1} n_{d=j} (1 - p(1 - p)^j) \\ &= \sum_{j=0}^{N-1} \Pr[d = j] (1 - p(1 - p)^j). \end{aligned} \quad (29)$$

Using the definition of pgf of the node degree (17), we obtain:

$$P_{AM} = \sum_{j=0}^{N-1} \Pr[d=j] \left(1 - p(1-p)^j\right) = 1 - p\varphi_D(1-p) \quad (30)$$

By raising P_{AM} to the N th power, the arithmetic stochastic approximation is derived as:

$$\overline{\text{nRel}}_G^{\text{arith}}(p) = (1 - p\varphi_D(1-p))^N \quad (31)$$

The geometric mean of $1 - f_i$ over all nodes is $P_{GM} = \sqrt[N]{\prod_{i=1}^N (1 - f_i)}$, and by the definition of f_i :

$$P_{GM} = \sqrt[N]{\prod_{i=1}^N (1 - p(1-p)^{d_i})}. \quad (32)$$

Since the arithmetic mean is always larger than or equal to the geometric mean, it holds that $\overline{\text{nRel}}_G^{\text{arith}}(p) \geq \overline{\text{nRel}}_G^{\text{geom}}(p)$.

The computational complexity of the arithmetic stochastic approximation, $\overline{\text{nRel}}_G^{\text{arith}}(p) = \exp(N \log(1 - p\varphi_D(1-p)))$, depends on the computation of $\varphi_D(1-p)$, which has a complexity of $o(\|D\|)$, where $\|D\|$ represents the number of distinct degrees in the graph. In contrast, the geometric stochastic approximation, $\overline{\text{nRel}}_G^{\text{geom}}(p)$, involves calculating the term $(1 - p(1-p)^{d_i})$ for each of the N nodes, resulting in a computational complexity of $o(N)$. Therefore, in networks where $\|D\| \ll N$, the arithmetic stochastic approximation requires significantly fewer computational resources, leading to a notable reduction in computation time and increased efficiency. As the network size increases, this advantage becomes even more significant, making the arithmetic stochastic approximation a practical choice for analyzing large-scale networks with a limited number of distinct degrees. For cases requiring higher precision, the geometric stochastic approximation $\overline{\text{nRel}}_G^{\text{geom}}(p)$ can be used, while the arithmetic stochastic approximation $\overline{\text{nRel}}_G^{\text{arith}}(p)$ is preferred for its computational efficiency.

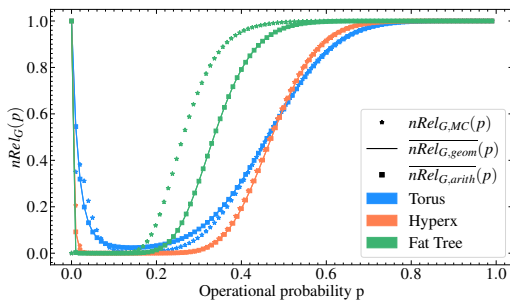


Fig. 11: Arithmetic Stochastic approximation $\overline{\text{nRel}}_G^{\text{arith}}(p)$, geometric stochastic approximation $\overline{\text{nRel}}_G^{\text{geom}}(p)$ and Monte Carlo simulations ($M = 10000$) for the node reliability of three structured network topologies: 3D Torus ($6 \times 6 \times 6$), HyperX (4D, 4 switches per dimension), and k -ary Fat-Tree ($k = 24$).

B. Limitations of the stochastic approximations

Fig. 9 and 11 show that all three stochastic approximations match the Monte-Carlo benchmark for the *Torus* and *HyperX* networks but clearly underestimate the node reliability of the *Fat-Tree*. The discrepancy comes from treating the events X_i ($i = 1, \dots, N$) as independent. In large, moderate-degree graphs each event involves (almost) disjoint neighbour sets, so the independence product is close to truth. In a *Fat-Tree* many leaves share the same aggregation and core switches; the events then overlap and the product formula is too small.

The example network in Fig. 12 illustrates this overlap for two nodes that share neighbours: the exact joint probability $\Pr[X_i \cap X_j]$ exceeds the independence product $\Pr[X_i] \Pr[X_j]$. The same mechanism, amplified over thousands of leaves, explains the gap observed on the *Fat-Tree*.

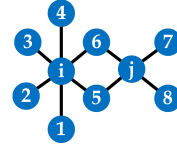


Fig. 12: Illustration of dependence between events X_i and X_j . Each node is independently work with probability p . The probability of event $X_i = \{\text{Node } i \text{ fails}\} \cup \{\text{Node } i \text{ is active and at least one neighbor of } i \text{ is active}\}$ is $\Pr[X_i] = 1 - p(1-p)^6$. For node j the probability is $\Pr[X_j] = 1 - p(1-p)^4$. The joint probability is $\Pr[X_i \cap X_j] = 1 - p(1-p)^6 - p(1-p)^4 + p^2(1-p)^8$. If the two events were independent, the probability is $\Pr[X_i \cap X_j] = \Pr[X_i] \Pr[X_j] = 1 - p(1-p)^6 - p(1-p)^4 + p^2(1-p)^{10}$.

V. ANALYTICAL OPTIMIZATION STRATEGY FOR THE NODE-RELIABILITY-BASED k -GRIP PROBLEM

The reliability is an important robustness measure of graph $G(V, E)$. Given a connected graph $G = (V, E)$ and a budget of k links to be added, a set $\mathcal{S} \subset \binom{N}{2} \setminus L$ of size k that optimizes the robustness of G is a common problem. Predari *et al.* refer to this optimisation problem as k -GRIP problem, short for graph robustness improvement problem [33]. We select the reliability polynomial $\text{Rel}_G(p)$ (node reliability polynomial $\text{nRel}_G(p)$) as robustness measure for k -GRIP. Within the stochastic approximations of reliability polynomial $\text{Rel}_G(p)$ and node reliability polynomial $\text{nRel}_G(p)$, we give optimal solutions $\mathcal{S}$ of $\text{Rel}_G(p)$ and $\text{nRel}_G(p)$. The optimal solution $\mathcal{S}$ are approximate optimal solutions for reliability-based k -GRIP and node-reliability-based k -GRIP problems.

By using the reliability polynomial as an objective measure, we design networks that optimize for specific reliability criteria. For example, to enhance a network's reliability, we might seek to add or reinforce connections (links) in such a way that the reliability polynomial achieves its highest possible values under expected operational probabilities. Since computing the exact expression of the reliability polynomial is NP-hard, it is not possible to provide an analytical solution to the reliability-based k -GRIP problem. However, a stochastic approximation of the reliability polynomial can be optimized analytically,

which offers a practical approach to solving the reliability-based k -GRIP problem.

We recall the stochastic approximation of the reliability polynomial in Eq. (24),

$$\overline{\text{Rel}}_G(p) = (1 - \varphi_D(1 - p))^N,$$

and the stochastic approximation of the node reliability polynomial in Eq. (23),

$$\overline{\text{nRel}}_G(p) = (1 - \varphi_D(1 - p))^{Np},$$

that both depend on the value of term $1 - \varphi_D(1 - p)$. The function $f(x) = x^c$, where c is a positive number, is monotonically increasing for x in domain $[0, 1]$. Thus a larger $1 - \varphi_D(1 - p)$ contributes to a higher reliability and node reliability. Consequently, the problem of optimizing the stochastic approximations $\overline{\text{Rel}}_G(p)$ and $\overline{\text{nRel}}_G(p)$ reduces to maximizing the value of $1 - \varphi_D(1 - p)$. We denote the graph obtained by adding the links of $\mathcal{S}$ into G as $G' := G \cup \mathcal{S}$ and the degree distribution of graph G' as D' , where $\mathcal{S} \subset \binom{N}{2} \setminus L$. Based on the analysis in our previous work [25], the term $1 - \varphi_D(1 - p)$ can be expressed as

$$1 - \varphi_D(1 - p) = \frac{1}{N} \sum_{i=1}^N \left(1 - (1 - p)^{d_i}\right) \quad (33)$$

where d_i is the degree of node i . Here we denote the degree vector of graph G as $\mathbf{d} = [d_1, d_2, \dots, d_N]$ and the degree change vector after k links are added into G as $\mathbf{a} = [a_1, a_2, \dots, a_N]$, where $a_i \geq 0$. Then the degree vector of graph G' becomes $\mathbf{d}' = [d_1 + a_1, d_2 + a_2, \dots, d_N + a_N]$. The reliability-based k -GRIP problem is then transformed into:

Objective:

$$\begin{aligned} & \max_{\mathbf{a}} 1 - \varphi_{D+\mathbf{a}}(1 - p) \\ &= \max_{\mathbf{a}=[a_1, a_2, \dots, a_N]} \sum_{i=1}^N \left(1 - (1 - p)^{d_i + a_i}\right) \end{aligned} \quad (34)$$

Subject to:

$$\text{s.t. } \sum_{i=1}^N a_i = 2k, a_i \geq 0, a_i \in \mathbb{Z} \quad (35)$$

Suppose there are two sets

$$\begin{aligned} \mathbf{a}_1 &= [a_1, a_2, \dots, a_m, \dots, a_n, \dots, a_N] \\ \mathbf{a}_2 &= [a_1, a_2, \dots, a_m + 1, \dots, a_n - 1, \dots, a_N] \end{aligned} \quad (36)$$

where $a_m, a_n \geq 1$.

The only difference between $\mathbf{a}_1$ and $\mathbf{a}_2$ is that the m th element of $\mathbf{a}_2$ is the m th element of $\mathbf{a}_1$ plus one, and the n th element of $\mathbf{a}_2$ is the n th element of $\mathbf{a}_1$ minus one. The difference of $\sum_{i=1}^N \left(1 - (1 - p)^{d_i + a_i}\right) |_{\mathbf{a}_2}$ and $\sum_{i=1}^N \left(1 - (1 - p)^{d_i + a_i}\right) |_{\mathbf{a}_1}$ is

$$\begin{aligned} \Delta &= \sum_{i=1}^N \left(1 - (1 - p)^{d_i + a_i}\right) |_{\mathbf{a}_2} - \sum_{i=1}^N \left(1 - (1 - p)^{d_i + a_i}\right) |_{\mathbf{a}_1} \\ &= p \left((1 - p)^{d_n + a_n - 1} - (1 - p)^{d_m + a_m} \right) \end{aligned} \quad (37)$$

The value of Δ is larger than 0 only when $d_n + a_n - 1 > d_m + a_m$. Here $d_m + a_m$ and $d_n + a_n$ are the degree of node m and node n after k links are added to the graph according to set $\mathbf{a}_1$. When the degree $d_n + a_n$ of the node n is larger than $d_m + a_m$ of node m plus one, reconnecting the end of one link connect to node n to node m contribute to a better link-adding set $\mathcal{S}$. Here we define a k -GRIP descending restructuring, which disconnects the end of an added link to a node n with degree $d_n + a_n$ in the graph G' and reconnects that end of the link to another node m with degree $d_m + a_m$, where $d_n + a_n - 1 > d_m + a_m$ and $a_n > 0$. The ascending restructuring is defined as the inverse of the descending restructuring. The above analysis shows that $1 - \varphi_{D+\mathbf{a}}(1 - p)$ after descending restructuring is always larger than that before descending restructuring.

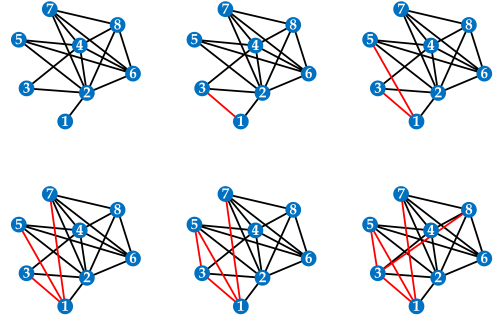


Fig. 13: Schematic illustration of the **Greedy-LD** algorithm, which iteratively adds 5 links (shown in red) to a graph with $N = 8$ nodes and $L = 16$ links. Each panel shows the intermediate graph after one additional link is added, highlighting how the algorithm prioritizes nodes with the lowest degree at each step.

We denote the set of all possible graphs, where k links are added into graph G , as $\langle G \rangle$. Start from a random graph G' in $\langle G \rangle$, any graph in $\langle G \rangle$ can be obtained by multiple descending and ascending restructurings. Since descending restructuring always contribute to a higher $\sum_{i=1}^N \left(1 - (1 - p)^{d_i + a_i}\right)$, the optimal graph G^* is a graph where no descending restructuring could occur. To construct the optimal graph G^* , links can be greedily added by connecting pairs of nodes with the lowest degrees, provided the link does not already exist. In this paper, the algorithm of greedily adding k links between pairs of nodes with the lowest degrees that are not already connected is referred to as the **Greedy Lowest-Degree Pairing Link Addition (Greedy-LD) Algorithm**.

With a bucket list that stores nodes by current degree, the initial degree scan costs $\mathcal{O}(L)$ time, where L is the number of links in the graph. Each of the k link insertions then requires only constant time for bucket updates and an $\mathcal{O}(1)$ adjacency check, giving an overall running time of $\mathcal{O}(L + k)$. Fig. 14 shows the distribution of the average node reliability over $p \in [0, 1]$ for all possible link addition configurations, where 5 links are added to base graphs with $N = 7, 8, 9$ nodes and $L = 11, 16, 24$ links, respectively. In all three cases, the **Greedy-LD** algorithm yields results that rank within the top 2% of all configurations, indicating that it consistently produces high-reliability structures with

Algorithm 1 Greedy Link Addition Based on the Minimum Degree–Sum

Input: connected graph $G = (V, E)$, number of links to add k

Output: augmented graph G^*

```

1: compute initial degrees  $\{d_v \mid v \in V\}$ 
2: for  $t \leftarrow 1$  to  $k$  do
  (a) score every missing link
3:   for all  $\{u, v\} \subset V$  s.t.  $(u, v) \notin E$  do
4:      $s(u, v) \leftarrow d_u + d_v$   $\triangleright$  degree sum of the pair
5:   end for
  (b) identify best pairs
6:    $s_{\min} \leftarrow \min_{\{u, v\} \notin E} s(u, v)$ 
7:    $C \leftarrow \{\{u, v\} \notin E \mid s(u, v) = s_{\min}\}$   $\triangleright$  all pairs with
    the smallest degree sum
  (c) random tie-break
8:   uniformly pick one link  $\{i, j\}$  from  $C$ 
  (d) update graph and degrees
9:    $E \leftarrow E \cup \{(i, j)\}$ ;  $d_i \leftarrow d_i + 1$ ;  $d_j \leftarrow d_j + 1$ 
10: end for
11: return  $G^* = (V, E)$ 

```

simple computation. To further evaluate the performance of the **Greedy-LD Algorithm** in improving network reliability, we applied **Greedy-LD Algorithm** and other two link-adding strategies to real-world networks from the Network Repository [34] and compared their effectiveness. The first strategy, **Random Pairing Strategy**, adds links randomly between node pairs that are not already linked. The second strategy, **Greedy Highest-Degree Pairing Strategy**, focuses on adding links between the highest-degree nodes that are not yet connected, aiming to strengthen the already well-connected nodes. Figures in Fig. 15 illustrate that **Greedy-LD Algorithm** has the most significant effect in enhancing network reliability and node reliability, far outperforming random link addition and adding links between high-degree nodes. Adding links between high-degree nodes shows the worst performance, with almost no noticeable impact on network reliability. The simulation demonstrates that one of the most effective ways to enhance network robustness from the perspective of network connectivity is to add links between low-degree nodes.

VI. CONCLUSION

Calculating the exact node-reliability polynomial is known to be NP-hard, prompting the need for efficient approximation techniques. The node-reliability polynomial of a graph has been approximated by Laplace’s method. For the node-reliability polynomial, utilizing “the C -form” and “ S -form” representations. We demonstrated that, by approximating these forms with probabilistic methods, significant computational efficiency can be achieved, while maintaining reasonable accuracy when approximating the node-reliability polynomial $nRel_G(p)$.

We proposed two simulation approaches: a standard Monte-Carlo estimator and a Laplace–Monte-Carlo variant. Because the Laplace term already captures the main behaviour

of $nRel(p)$, the Laplace–Monte-Carlo method requires far fewer samples and is therefore computationally lighter while achieving the same level of accuracy as the plain Monte-Carlo approach. Additionally, we introduced three degree-based stochastic approximations—Laplace, arithmetic, and geometric—leveraging the probability-generating function of node degrees. These stochastic approximations provide quick and reasonably accurate estimates, particularly effective for large and dense networks.

Finally, we introduced a Greedy Lowest-Degree Pairing Link Addition (**Greedy-LD**) algorithm that simply connects pairs of nodes with the smallest current degree. Despite its simplicity and $\mathcal{O}(k + L)$ time complexity, Greedy-LD consistently ranks within the top 2% of all possible link-addition configurations in our experiments, and provides larger reliability gains than both random link insertion and highest-degree pairing strategies.

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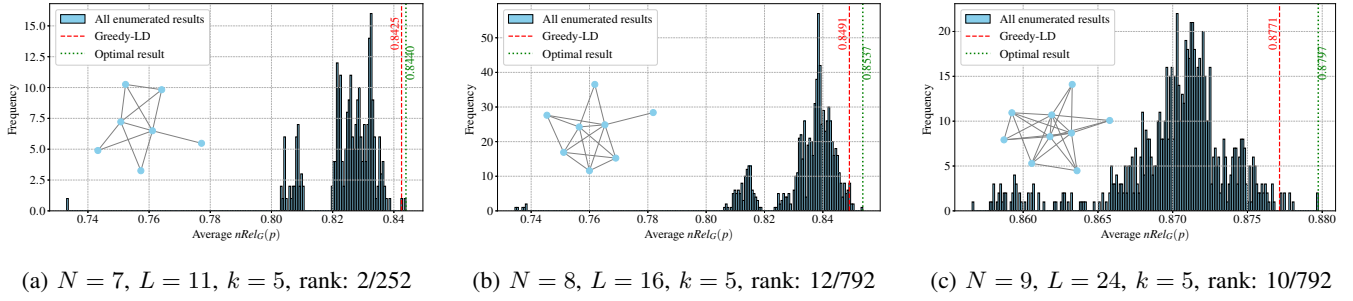


Fig. 14: Distribution of the average node reliability polynomial over $p \in [0, 1]$ for all possible configurations obtained by adding 5 links to a base graph with: (a) $N = 7, L = 11$, (b) $N = 8, L = 16$, and (c) $N = 9, L = 24$ links. The red dashed line shows the result of the **Greedy-LD** algorithm, while the green dotted line marks the optimal reliability among all configurations. In these three cases, the greedy strategy ranks 2nd out of 252, 12th out of 792, and 10th out of 792, respectively, demonstrating consistently near-optimal performance across different network sizes.

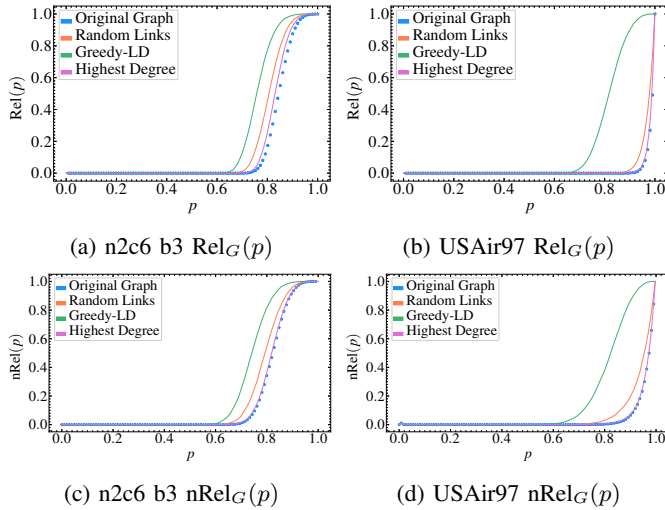


Fig. 15: Monte Carlo simulations results ($M = 10000$) of reliability polynomial $\text{Rel}_G(p)$ and node reliability polynomial $\text{nRel}_G(p)$ in two real-world graphs and graphs constructed by adding links to these graphs for three different strategies. ‘n2c6 b3’ is a simplicial complex network with $N = 1365$ nodes and $L = 5263$ links [34]. $l = 500$ links are added into the ‘n2c6 b3’ graph for three different strategies. ‘USAir97’ is the 1997 U.S. flight network with $N = 332$ nodes and $L = 2126$ links [34]. $l = 100$ links are added into the ‘USAir97’ graph in three different strategies. (a) Reliability polynomial $\text{Rel}_G(p)$. (b) Reliability polynomial $\text{Rel}_G(p)$. (c) Node reliability polynomial $\text{nRel}_G(p)$. (d) Node reliability polynomial $\text{nRel}_G(p)$.

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APPENDIX A SYMBOLS

Only when explicitly mentioned, we will deviate from the symbols outlined here.

General Graph Notation

$G = (V, E)$	Graph with node set V and link set E
$N = V $	Number of nodes in the graph
$L = E $	Number of links in the graph
p	Probability that each node is active (operational probability)
$\mathcal{S}$	Set of links to be added (in k -GRIP)
$G' = G \cup \mathcal{S}$	Graph after adding links in $\mathcal{S}$

Node Reliability and Related Functions

$\text{nRel}_G(p)$	Node reliability polynomial of graph G
$\text{Rel}_G(p)$	Classical reliability polynomial (link-based)
$C_j(G)$	Number of size- j vertex cut sets
$S_k(G)$	Number of connected subgraphs with k nodes
$c_j(G)$	Probability that a size- j node set disconnects G
$s_k(G)$	Probability that a size- k node set induces a connected subgraph
$\tilde{c}_j(G)$	Monte Carlo estimate of $c_j(G)$
$\hat{G}_k$	Residual graph after $N - k$ nodes are removed

Monte Carlo and Stochastic Approximations

M	Number of Monte Carlo trials
$\text{nRel}_{G, \text{MC}}(p)$	Monte Carlo estimate of $\text{nRel}_G(p)$
$\overline{\text{nRel}}_G(p)$	Stochastic approximation of $\text{nRel}_G(p)$
$\text{nRel}_G^{\text{Lap}}(p)$	Laplace approximation of $\text{nRel}_G(p)$
R_j	Number of trials where removal of j nodes disconnects the graph

Probabilistic and Degree-related Terms

D	Degree of a randomly selected node
$D_{\min}$	Minimum degree of the graph G
$\hat{D}_{\min}$	Minimum degree in the residual graph $\hat{G}_k$
$\varphi_D(z)$	Probability generating function of the degree distribution D
$o(\cdot)$	Asymptotic notations
$\delta(x - \mu)$	Dirac delta function centered at μ

Matrix Formulations for Cut-set Inference

$\mathbf{P}$	Vandermonde-like matrix with $(1 - p_i)^j p_i^{N-j}$ entries
$\mathbf{C}$	Cut-set count vector $[C_1, \dots, C_N]^T$
$\mathbf{r}$	Vector $[\text{nRel}_G(p_1), \dots, \text{nRel}_G(p_N)]^T$

Link Addition and k -GRIP Optimisation

k	Number of links to be added
$\mathbf{a}$	Degree increment vector $[a_1, \dots, a_N]$
$\mathbf{d}$	Original degree vector of graph G
Δ	Gain in objective value from rewiring one link
$\langle G \rangle$	Set of graphs obtained by adding k links to G

Error Metrics and Evaluation

MSE	Mean squared error
MAE	Mean absolute error
Max Error	Maximum error between approximation and ground truth

APPENDIX B

ERROR ANALYSIS OF THE LAPLACE APPROXIMATION

The error of the Laplace approximation arises from two main sources: the normal approximation of the Binomial PMF in Eq. (7), and the Dirac delta approximation in Eq. (11).

The function $s_{Nx}(G)$ is obtained by linearly interpolating the discrete sequence $\{s_k(G)\}_{k=0}^N$, with each $s_k(G) \in [0, 1]$. In particular, if a piecewise linear interpolation is used, then $s_{Nx}(G) = s_k + [s_{k+1} - s_k] \left(x - \frac{k}{N}\right) N$, which implies a Lipschitz constant $L = N \max_k |s_{k+1} - s_k| \leq N$, since $|s_{k+1} - s_k| \leq 1$. Hence $s_{Nx}(G)$ satisfies Lipschitz continuity, and thus its second derivative, $s''_{Nx}(G)$, is bounded.

1) *Error $\epsilon_{N,p}$ from the Normal Approximation of the Binomial PMF:* Let $k \sim \text{Bin}(N, p)$ and define $Z = \frac{k - Np}{\sqrt{Np(1-p)}}$. Set $\Delta_k := \binom{N}{k} p^k (1-p)^{N-k} - \frac{\Phi(Z)}{\sqrt{Np(1-p)}}$, where $\Phi(\cdot)$ denotes the standard normal pdf at Z . By the De Moivre–Laplace local limit theorem, for $|k - Np| \leq c\sqrt{N}$ we have $\Delta_k = o\left(\frac{\Phi(Z)}{\sqrt{Np(1-p)}}. Hence $\sum_{k=0}^N |\Delta_k| = o\left(\frac{1}{\sqrt{Np(1-p)}}. For a weight $s_k(G)$ with $\max_k |s_k(G)| \leq 1$, define$$

$$\epsilon_{N,p} = \left| \sum_{k=0}^N s_k(G) \Delta_k \right| \leq \sum_{k=0}^N |\Delta_k| = o\left(\frac{1}{\sqrt{Np(1-p)}}$$

which shows that the error from the normal approximation is $\epsilon_{N,p} = o\left(\frac{1}{\sqrt{Np(1-p)}}.$

2) *Error from the Dirac Delta Approximation:* Because the Gaussian kernel becomes sharply concentrated near $x = p$ as $\tilde{\sigma} \rightarrow 0$, we expand $s_{Nx}(G)$ in a Taylor series around $x = p$:

$$\begin{aligned} s_{Nx}(G)|_{x=p} &= s_{Np}(G) + (x - p) \frac{d}{dx} s_{Nx}(G)|_{x=p} \\ &\quad + \frac{1}{2} (x - p)^2 \frac{d^2}{dx^2} s_{Nx}(G)|_{x=p} + R_3(x), \end{aligned} \quad (38)$$

where $R_3(x)$ denotes the third- and higher-order remainder terms. Substituting into the integral in Eq. (9) yields

$$\begin{aligned} \text{nRel}_G(p) &\simeq \int_0^1 \left[s_{Np}(G) + (x - p) s'_{Np}(G) \right. \\ &\quad \left. + \frac{1}{2} (x - p)^2 s''_{Np}(G) + R_3(x) \right] \phi(x) dx \\ &= s_{Np}(G) + \frac{1}{2} s''_{Np}(G) \tilde{\sigma}^2 + \mathcal{O}(\tilde{\sigma}^3), \end{aligned} \quad (39)$$

where $\phi(x)$ is the normalized Gaussian kernel centered at $x = p$. Here, the zeroth-order term $\int_0^1 s_{Np}(G) \phi(x) dx$ integrates to $s_{Np}(G)$, the first-order term vanishes because $\int_0^1 (x - p) \phi(x) dx = 0$, and the second-order term $\int_0^1 \frac{1}{2} (x - p)^2 s''_{Np}(G) \phi(x) dx$ yields $\frac{1}{2} s''_{Np}(G) \tilde{\sigma}^2$, since $\int_0^1 (x - p)^2 \phi(x) dx = \tilde{\sigma}^2$. The remainder $R_3(x)$ contributes $\mathcal{O}(\tilde{\sigma}^3)$, as $(x - p)^3 \phi(x)$ integrates to $\mathcal{O}(\tilde{\sigma}^3)$.

Hence we obtain

$$\text{nRel}_G(p) = s_{Np}(G) + o\left(\frac{1}{\sqrt{Np(1-p)}}\right) + o(s''_{Np}(G)\tilde{\sigma}^2), \quad (40)$$

where $\tilde{\sigma}^2 = \frac{p(1-p)}{N}$ and $s''_{Np}(G)$ is bounded. Thus, the error term $o(s''_{Np}(G)\tilde{\sigma}^2)$ is effectively $o(\frac{1}{N})$, since $s''_{Np}(G)$ remains finite and $\tilde{\sigma}^2 = \frac{p(1-p)}{N}$. In other words, Eq. (40) shows that the overall error of the Laplace approximation includes a component $o(\frac{1}{\sqrt{Np(1-p)}})$ from the Binomial-to-normal step and an additional $o(\frac{1}{N})$ from higher-order corrections in the Dirac delta approximation.

APPENDIX C

ERROR ANALYSIS OF THE MONTE CARLO METHOD

The exact node reliability polynomial $\text{nRel}_G(p)$ is $\text{nRel}_G(p) = 1 - \sum_{j=1}^N \binom{N}{j} c_j(G) p^{N-j} (1-p)^j$, where $c_j(G) = C_j(G)/\binom{N}{j}$ is the probability that the network becomes disconnected upon the random failure of j nodes. Let $\tilde{C}_j(G)$ denote the number of trials in which the network becomes disconnected, then the estimator is $\tilde{c}_j(G) := \frac{\tilde{C}_j(G)}{M}$. The Monte Carlo approximation of the node reliability polynomial is $\text{nRel}_{G,\text{MC}}(p) = 1 - \sum_{j=1}^N \binom{N}{j} \tilde{c}_j(G) p^{N-j} (1-p)^j$.

Each $\tilde{c}_j(G)$ is an unbiased estimator of $c_j(G)$, with variance

$$\text{Var}[\tilde{c}_j(G)] = \frac{c_j(G)(1-c_j(G))}{M}.$$

Let $a_j := \binom{N}{j} p^{N-j} (1-p)^j$ be the weight of each term. Then the variance of the estimator is

$$\begin{aligned} \text{Var}[\text{nRel}_{G,\text{MC}}(p)] &= \sum_{j=1}^N a_j^2 \text{Var}[\tilde{c}_j(G)] \\ &= \frac{1}{M} \sum_{j=1}^N a_j^2 c_j(G)(1-c_j(G)) \\ &\leq \frac{1}{4M} \sum_{j=1}^N a_j^2. \end{aligned} \quad (41)$$

Hence, for fixed N and p , the *mean squared error* (MSE) satisfies $\text{MSE}[\text{nRel}_{G,\text{MC}}(p)] = o(\frac{1}{M})$. The *mean absolute error* (MAE) is then of order $o(\frac{1}{\sqrt{M}})$. To bound the probability of large deviations from the true value, we apply Hoeffding's inequality. Let $X_{j,i} \in \{0,1\}$ be the outcome of the i -th trial for $c_j(G)$, and define $Y_{j,i} := a_j(X_{j,i} - c_j(G))$. Then

$$\text{nRel}_{G,\text{MC}}(p) - \text{nRel}_G(p) = \frac{1}{M} \sum_{j=1}^N \sum_{i=1}^M Y_{j,i},$$

where each $Y_{j,i} \in [-a_j, a_j]$ is independent and has zero mean. Hoeffding's inequality gives: $\Pr(|\text{nRel}_{G,\text{MC}}(p) - \text{nRel}_G(p)| > \epsilon) \leq 2 \exp\left(-\frac{2M\epsilon^2}{\sum_{j=1}^N a_j^2}\right)$.

The sum $\sum_{j=0}^N a_j^2$ corresponds to the squared binomial PMF:

$$\sum_{j=0}^N \left[\binom{N}{j} p^{N-j} (1-p)^j \right]^2,$$

which equals the probability that two independent $\text{Bin}(N, 1-p)$ variables take the same value. For large N , this sum satisfies

$$\sum_{j=0}^N a_j^2 = \mathcal{O}\left(\frac{1}{\sqrt{Np(1-p)}}\right),$$

and is maximized at $p = 0.5$, where it evaluates to $\binom{2N}{N}/4^N$.

Hence, the probability of large Monte Carlo estimation error decays exponentially in M , and overall error satisfies $\Pr(|\text{nRel}_{G,\text{MC}}(p) - \text{nRel}_G(p)| > \epsilon) \leq 2 \exp\left(-CM\epsilon^2\sqrt{Np(1-p)}\right)$, for some constant $C > 0$. This result ensures that the Monte Carlo estimator becomes increasingly accurate as the number of simulations M increases.

APPENDIX D

NODE RELIABILITY POLYNOMIALS FOR SOME GRAPH FAMILIES

A. Complete graph K_N

For a complete graph K_N with N nodes, the only way to disconnect the graph is when all the nodes fail. For a node operational probability p , the probability that all nodes are failed is $(1-p)^N$. Thus we obtain the following node reliability polynomial:

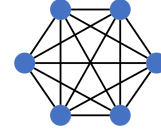


Fig. 16: Complete graph K_N

$$\text{nRel}_G(p) = \Pr[\hat{G} \text{ is connected}] = 1 - (1-p)^N \quad (42)$$

B. Complete graph K_{N-1} with a pendant node

We will denote the complete graph K_{N-1} with a pendant node as K_N^* .

For this graph, which contains N nodes, if the pendant node fails, then the probability that the residual part is connected equals to the probability that a complete graph K_{N-1} is connected. If the pendant node is operational, then residual graph $\hat{G}$ is connected under one of two conditions: 1. The edge node connected to the pendant node is operational. 2. The edge node connected to the pendant node is not operational and all other nodes in K_{N-1} have also failed. Hence we have for the node reliability polynomial of K_N^* :

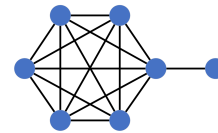


Fig. 17: The Graph $K_{N-1} + P_1$

$$\begin{aligned} \text{nRel}_G(p) &= \Pr[\hat{G} \text{ is connected}] \\ &= (1-p)(1-(1-p)^{N-1}) + p^2 + p(1-p)^{N-1} \end{aligned} \quad (43)$$

C. Cycle graph C_N

For a cycle graph C_N with N nodes, after k nodes are removed, the remaining $N - k$ nodes should be connected and for every $k \in [1, N - 1]$, there are N different residual connected graphs on $N - k$ nodes. Thus the probability that the residual graph $\hat{G}$ is connected is the probability that all nodes are functional plus the probability that k nodes fail but $\hat{G}$ is still connected, leading to

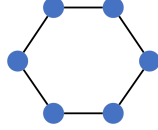


Fig. 18: Cycle graph C_N

$$\begin{aligned}
 \text{nRel}_G(p) &= \Pr[\hat{G} \text{ is connected}] \\
 &= p^N + N \sum_{k=1}^{N-1} p^{N-k} (1-p)^k \\
 &= N \sum_{k=0}^{N-1} p^{N-k} (1-p)^k - (N-1)p^N \\
 &= \frac{Np(p^N - (1-p)^N)}{2p-1} - (N-1)p^N \quad (44)
 \end{aligned}$$

Noting that at $p = 0.5$, the node reliability polynomial $\text{nRel}_G(p) = \frac{Np(p^N - (1-p)^N) - (2p-1)(N-1)p^N}{2p-1}$ get a 0/0 value. Applying L'Hôpital's Rule, we differentiate the numerator and denominator with respect to p separately. The denominator derivative is simply 2, while the numerator derivative is $\frac{d}{dp} [Np(p^N - (1-p)^N) - (2p-1)(N-1)p^N] = \frac{N^2 - N + 1}{2^{N-1}}$. Thus, the limiting value is $\lim_{p \rightarrow 0.5} \text{nRel}_G(p) = \frac{N^2 - N + 1}{2^N}$. Because the node reliability polynomial $\text{nRel}_G(p)$ is continuous and differentiable, the value of the node reliability polynomial $\text{nRel}_G(p)$ at $p = 0.5$ is equal to $\lim_{p \rightarrow 0.5} \text{nRel}_G(p) = \frac{N^2 - N + 1}{2^N}$.

D. Path graph P_N

For a path graph P_N , if we remove i ($i \in [0, j]$) connected edge nodes from the left side of the path graph and $j - i$ connected edge nodes from the right side, the graph still connected. Thus the number of vertex cut sets C_j equals $\binom{N}{j} - j - 1$. Therefore for the node reliability polynomial of path graph P_N it holds:

$$\begin{aligned}
 \text{nRel}_G(p) &= \Pr[\hat{G} \text{ is connected}] \\
 &= 1 - \sum_{j=0}^N \left(\binom{N}{j} - j - 1 \right) p^{N-j} (1-p)^j \\
 &= \sum_{j=0}^N (j+1) p^{N-j} (1-p)^j \\
 &= p^N \sum_{j=0}^N (j+1) \left(\frac{1-p}{p} \right)^j \\
 &= -p^{N-2} \sum_{j=0}^N \left(-\frac{(j+1)}{p^2} \left(\frac{1-p}{p} \right)^j \right) \quad (45)
 \end{aligned}$$

Because the derivative of $f(z) = \left(\frac{1-z}{z} \right)^{j+1}$ is $\frac{d}{dz} \left(\frac{1-z}{z} \right)^{j+1} = -\frac{(j+1)}{z^2} \left(\frac{1-z}{z} \right)^j$, then we have

$$\begin{aligned}
 \text{nRel}_G(p) &= -p^{N-2} \sum_{j=0}^N \left(-\frac{(j+1)}{p^2} \left(\frac{1-p}{p} \right)^j \right) \\
 &= -p^{N-2} \sum_{j=0}^N \frac{d}{dp} \left(\frac{1-p}{p} \right)^{j+1} \\
 &= -p^{N-2} \frac{d}{dp} \sum_{j=0}^N \left(\frac{1-p}{p} \right)^{j+1} \\
 &= -p^{N-2} \frac{d}{dp} \left(\frac{1-p}{2p-1} \left(1 - \left(\frac{1-p}{p} \right)^{N+1} \right) \right) \\
 &= \frac{Np(1-p)^{N+1} - (N+1)p^2(1-p)^N + p^{N+2}}{(1-2p)^2} \quad (46)
 \end{aligned}$$



Fig. 19: Path graph P_N

Noting that at $p = 0.5$, the expression

$$\text{nRel}_G(p) = \frac{Np(1-p)^{N+1} - (N+1)p^2(1-p)^N + p^{N+2}}{(1-2p)^2}$$

results in an indeterminate form of type 0/0. Applying L'Hôpital's Rule twice, we differentiate both numerator and denominator twice with respect to p . The second derivative of the denominator is constant:

$$\frac{d^2}{dp^2} (1-2p)^2 = 8.$$

The second derivative of the numerator at $p = 0.5$ evaluates to $\frac{4N(N+1)}{2^N}$. Hence, the limiting value is

$$\lim_{p \rightarrow 0.5} \text{nRel}_G(p) = \frac{F''(0.5)}{8} = \frac{N(N+1)}{2^{N+1}}.$$

Because the node reliability polynomial $\text{nRel}_G(p)$ is continuous and differentiable, the value of the node reliability polynomial $\text{nRel}_G(p)$ at $p = 0.5$ is equal to $\lim_{p \rightarrow 0.5} \text{nRel}_G(p) = \frac{N(N+1)}{2^{N+1}}$.

E. Star graph S_N

For a star graph S_N with one central node and $N - 1$ edge nodes, as long as the central node is still operational, the residual graph $\hat{G}$ is connected. The other condition where the residual graph is still connected is where the central node has failed but only one edge node is operational. Therefore for the node reliability polynomial of star graph S_N it holds:

$$\begin{aligned} \text{nRel}_G(p) &= \Pr[\hat{G} \text{ is connected}] \\ &= p + (N - 1)p(1 - p)^{N-1} \end{aligned} \quad (47)$$

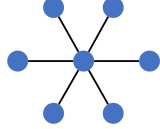


Fig. 20: Star graph S_N

F. Star graph S_{N-1} with a pendant node

We denote a star graph S_{N-1} with a pendant node by S_N^* . For this graphs, which contains N nodes, if the pendant node has failed, then the probability that the residual part is connected equals the probability that a star graph S_{N-1} is connected. If the pendant node is operational, the residual graph $\hat{G}$ is connected in three disjoint cases: 1. The edge node connected to the pendant node, and the central node are both operational. 2. The edge node connected to the pendant node is operational, all the other node fail. 3. Only the pendant node is operational.

Therefore, the node reliability polynomial of the graph S_N^* is given by:

$$\begin{aligned} \text{nRel}_G(p) &= \Pr[\hat{G} \text{ is connected}] \\ &= (1 - p) \left(p + (N - 2)p(1 - p)^{N-2} \right) \\ &\quad + p^3 + p^2(1 - p)^{N-2} + p(1 - p)^{N-1} \end{aligned} \quad (48)$$

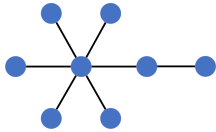


Fig. 21: The Graph $S_{N-1} + P_1$

APPENDIX E

THE NODE RELIABILITY POLYNOMIAL $\text{nRel}_G(p)$ OF ERDŐS-RÉNYI GRAPHS

In a Erdős-Rényi graph G_{ER} , a node connects to another node with a probability p_l . The random residual graph $\hat{G}_n$ of an Erdős-Rényi graph G_{ER} retains the Erdős-Rényi property, where residual nodes remain connected with probability p_l . Thus, the probability that the degree $\hat{D}_{\hat{G}_n}$ of a node in $\hat{G}_n$ equals 0 is:

$$\Pr[\hat{D}_{\hat{G}_n} = 0] = (1 - p_l)^{n-1} \quad (49)$$

Invoking the analysis in Section IV (23), the stochastic approximation of the node reliability polynomial of Erdős-Rényi graph is:

$$\overline{\text{nRel}}_{G,ER}(p) \simeq \Pr[\hat{D}_{\min} \geq 1] = (1 - (1 - p_l)^{Np-1})^{Np}. \quad (50)$$

Rewriting the exponent, we obtain:

$$\begin{aligned} \overline{\text{nRel}}_{G,ER}(p) &= (1 - (1 - p_l)^{Np-1})^{Np} \\ &= \left(1 - (1 - p_l)^{\frac{1}{p_l} p_l (Np-1)} \right)^{Np} \end{aligned} \quad (51)$$

Using $(1 - x)^{\frac{1}{x}} = e^{-1} + o(x)$ for x tends to 0 for small p_l , we have

$$\begin{aligned} \overline{\text{nRel}}_{G,ER}(p) &= \left(1 - (1 - p_l)^{\frac{1}{p_l} p_l (Np-1)} \right)^{Np} \\ &\simeq \left(1 - e^{-p_l(Np-1)} \right)^{Np} \end{aligned} \quad (52)$$

A new function of the node operational probability p is defined as:

$$b(p) = \frac{e^{N \cdot p \cdot p_l}}{N} \quad (53)$$

so that the link probability equals $p_l = \frac{(\log b(p)N)}{pN}$. Substituting $p_l = \frac{(\log b(p)N)}{pN}$ to (52)

$$\begin{aligned} \overline{\text{nRel}}_{G,ER}(p) &\simeq \left(1 - \exp \left(-\frac{\log(b(p)N)}{pN} \cdot (Np - 1) \right) \right)^{Np} \\ &\simeq \left(1 - \frac{1}{b(p)N} \right)^{Np} \end{aligned} \quad (54)$$

which for large N lead to

$$\overline{\text{nRel}}_{G,ER}(p) \simeq e^{-\frac{p}{b(p)}} \quad (55)$$

After substituting the definition $b(p) = \frac{e^{N \cdot p \cdot p_l}}{N}$ in (53) into (55), we obtain an approximation of the node reliability of Erdős-Rényi graph

$$\overline{\text{nRel}}_{G,ER}(p) \simeq \exp \left(-\frac{\bar{k}}{e^{\bar{k} \cdot p_l}} \right) \quad (56)$$

where $\bar{k} = Np$ is the average degree of the graph.

The degree distribution of the Erdős-Rényi graph follows a binomial distribution. The general approximation form 56 of the node reliability polynomial for an Erdős-Rényi graph ignores the specific degree distribution of the network; its value depends only on the number of nodes and the connection probability. The Law of Large Numbers states that, as the number of nodes N in an Erdős-Rényi graph increases, the degree of all nodes will converge to the expected value, $E[D_{ER}](N - 1)p$. The Central Limit Theorem further implies that as N grows larger, the degree distribution of the graph will increasingly resemble a Poisson distribution centered around the expected value $(N - 1)p$. Therefore, with a larger N , the degree distribution of a specific realization of an Erdős-Rényi graph will more closely match the theoretical degree distribution [23, p. 34-40]. Consequently, as the size of the Erdős-Rényi graph increases, the general form of its node

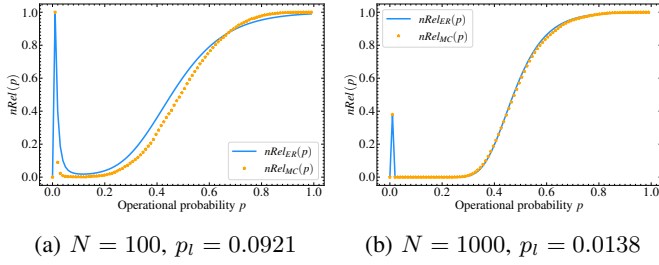


Fig. 22: Monte Carlo simulations $nRel_{G,MC}$ and general form of node reliability polynomials $nRel_{G,ER}(p)$ for Erdős-Rényi graphs with different number of nodes N and link probability p_l .

reliability polynomial becomes more accurate, as demonstrated in Fig. 22.

Studying the intersections of reliability polynomials helps evaluate and compare the reliability of different networks. Here we give an analytical solution of intersections $(p_i, nRel_G(p))$ of different Erdős-Rényi graphs. Given two different Erdős-Rényi graph G_1 (with N_1 nodes and link probability p_{l1}) and G_2 (with N_2 nodes and link probability p_{l2}), the intersection of the node reliability polynomials of these two graph is

$$p_i = \exp\left(\frac{\bar{k}_1 \log N_2 - \bar{k}_2 \log N_1}{\bar{k}_2 - \bar{k}_1}\right) \quad (57)$$

Invoking (55), the value of node reliability $nRel_G(p)$ of Erdős-Rényi graph is determined by the term $\frac{p}{b(p)}$. For two different Erdős-Rényi graph G_1 (with N_1 nodes and link probability p_{l1}) and G_2 (with N_2 nodes and link probability p_{l2}), the intersection of two reliability polynomial can be obtained by solving

$$\frac{p_i}{b_1(p_i)} = \frac{p_i}{b_2(p_i)} \quad (58)$$

computing p_i , $b_1(p_i)$ and $b_2(p_i)$ satisfy:

$$\begin{cases} p_i = \exp\left(\frac{\bar{k}_1 \log N_2 - \bar{k}_2 \log N_1}{\bar{k}_2 - \bar{k}_1}\right) \\ b_1(p_i) = \frac{e^{\bar{k}_1 p_i}}{N_1} \\ b_2(p_i) = \frac{e^{\bar{k}_2 p_i}}{N_2} \end{cases} \quad (59)$$

where $\bar{k}_1 = N_1 \cdot p_{l1}$, $\bar{k}_2 = N_2 \cdot p_{l2}$. Based on these p_i and $b_i(p_i)$, we obtain the intersection $(p_i, e^{-\frac{p_i}{b_1(p_i)}})$. Fig. 23 illustrates that as in our derivation in Eq.(59), the reliability polynomials of Erdős-Rényi graphs with same $p_i = 0.4$, $b(p_i) = 0.6$ intersect at point $(p_i, e^{-\frac{p_i}{b(p_i)}})$.

From (57), the condition for p_i to have a solution in the range $(0, 1)$ is:

- If $\bar{k}_2 > \bar{k}_1$, then $\frac{\log N_1}{\log N_2} > \frac{\bar{k}_1}{\bar{k}_2}$.
- If $\bar{k}_2 < \bar{k}_1$, then $\frac{\log N_1}{\log N_2} < \frac{\bar{k}_1}{\bar{k}_2}$.

which means that for two Erdős-Rényi graphs, the node reliability polynomials will only intersect if the graph with the smaller average degree has more nodes. For Erdős-Rényi graphs with same number of nodes N , the larger the average degree $\bar{k}$, the larger the node reliability polynomial $nRel_G(p)$ as is shown in Fig. 24. Fig. 24 also demonstrates that the larger

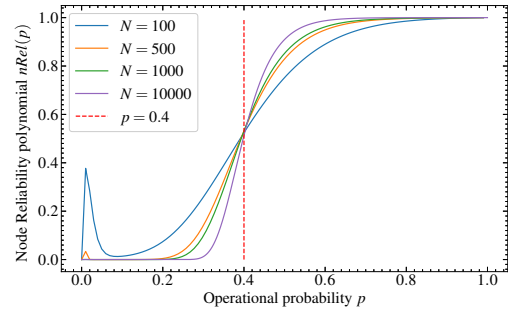


Fig. 23: The Monte Carlo simulation result of node reliability polynomials of Erdős-Rényi graphs with $p_i = 0.4$, different number of nodes N and auxiliary line $p = 0.4$.

the average degree $\bar{k}$, the faster the $nRel_G(p)$ curve changes from $0 + \epsilon$ to $1 - \epsilon$, where ϵ is a very small value. It is expected that for Erdős-Rényi graphs with very large average degree $\bar{k}$, the $nRel_G(p)$ will exhibit a sharp transition from 0 to 1, resembling a phase transition. Equation (55) shows that the node reliability polynomial of an Erdős-Rényi graph depends on $\frac{p}{b(p)}$:

$$\frac{p}{b(p)} = \frac{pN}{e^{N \cdot p \cdot p_l}} = c' \quad (60)$$

Taking the logarithm of both sides:

$$\log p + \log N - N \cdot p \cdot p_l = \log c' \quad (61)$$

For two values p_1 and p_2 , we have:

$$\log \frac{p_1}{p_2} - (p_1 - p_2)Np_l = \log \frac{c'_1}{c'_2} \quad (62)$$

Rearranging the terms:

$$p_1 - p_2 \approx -\frac{\log c'_1 - \log c'_2}{Np_l} \quad (63)$$

Thus, as the average degree $\bar{k} = Np_l$ becomes large, the range of p where $nRel_{G,ER}(p)$ changes from c_2 to c_1 is approximately on the order of $O\left(\frac{1}{Np_l}\right)$.

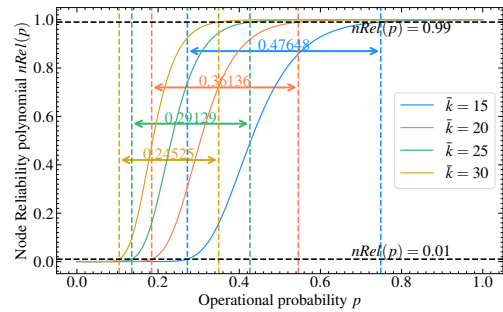


Fig. 24: the Monte Carlo simulation results of the node reliability polynomial $nRel_G(p)$ of Erdős-Rényi graphs with $N = 1000$ and different average degrees $\bar{k}$. The numbers in different colors indicate the length of the corresponding polynomials change from $nRel_G(p) = 0.01$ to $nRel_G(p) = 0.99$.

APPENDIX F

DERIVATION OF THE JOINT-CORRECTION FORMULA

Let X_i be the event that the i -th node is not in an isolated state, and set $f_i = \Pr[X_i]$. We wish to compute

$$P\left[\bigcap_{i=1}^N X_i\right] = E\left[\prod_{i=1}^N X_i\right].$$

A compact way to obtain the exact expansion (64) is via the multivariate moment-generating function (MGF) and cumulant-generating function (CGF) [1].

Define the MGF of $\{X_i\}$ by

$$M(t_1, \dots, t_N) = E\left[\exp\left(\sum_{i=1}^N t_i X_i\right)\right],$$

and the CGF by

$$K(t_1, \dots, t_N) = \log M(t_1, \dots, t_N).$$

By construction the mixed partial derivative

$$\kappa(X_{i_1}, \dots, X_{i_k}) = \left. \frac{\partial^k K}{\partial t_{i_1} \dots \partial t_{i_k}} \right|_{t=0}$$

is precisely the k -th order joint cumulant [1].

a) *Examples of cumulants.:*

- **First-order:** $\kappa(X_i) = E[X_i] = f_i$.
- **Second-order:** $\kappa(X_i, X_j) = E[(X_i - f_i)(X_j - f_j)] = \text{Cov}(X_i, X_j)$.

By the classical cumulant–moment formula (Leonov & Shiryayev [1], Proposition 2.3), we have

$$\Pr\left[\bigcap_{i=1}^N X_i\right] = \sum_{\pi \in \mathcal{P}([N])} \prod_{B \in \pi} \kappa(X_i : i \in B).$$

where $\mathcal{P}([N])$ denotes all set partitions of $\{1, \dots, N\}$. Pulling every singleton block's cumulant f_i outside the sum gives

$$\begin{aligned} \Pr\left[\bigcap_{i=1}^N X_i\right] &= \prod_{i=1}^N f_i \\ &\times \left\{ 1 + \sum_{k=2}^N \sum_{1 \leq i_1 < \dots < i_k \leq N} \frac{\kappa(X_{i_1}, \dots, X_{i_k})}{\prod_{m=1}^k f_{i_m}} \right\}. \end{aligned} \tag{64}$$

In the independent case all $\kappa(\cdot)$ for $k \geq 2$ vanish, recovering $\prod_i f_i$.

REFERENCES

- [1] G. Peccati and M. S. Taqqu, *Wiener Chaos: Moments, Cumulants and Diagrams: A survey with computer implementation*, vol. 1, Springer Science & Business Media, 2011.