

Temporal graph reproduction with RWIG

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Abstract—We examine the Random Walkers Induced temporal Graph (RWIG) model, which generates temporal graphs based on the co-location principle of M independent walkers that traverse the underlying Markov graph with different transition probabilities. Given the assumption that each random walker is in the steady state, we determine the steady-state vector $\bar{s}$ and the Markov transition matrix P_i of each walker w_i that can reproduce the observed temporal network $G_0, \dots, G_{K-1}$ with the lowest mean squared error. We also examine the performance of RWIG for periodic temporal graph sequences.

Index Terms—Temporal Networks, Generative Models, Network Dynamics, Markov Process, Random Walks, RWIG.

1 INTRODUCTION

TEMPORAL networks have attracted a lot of attention in the last decades [1], [2], [3]. Many complex systems, including infrastructural, biological, social, and financial networks, evolve over time which, in turn, affects the topology and the processes that propagate over these networks. Modeling these systems enhances our understanding of their dynamics and the processes that propagate through them, such as diffusion and contagion, while also providing valuable predictions [4], [5].

Modeling temporal networks is more difficult and challenging than modeling static graphs. While the static network generation is aimed to reproduce some emergent network structure (e.g. power-law degree distribution [6], community structure [7], [8], motifs [9] or small-world [10] properties), the temporal network generation should take into account a highly non-trivial interplay between the topology and the evolutionary process of the network, which exhibits temporal patterns, memory effects and dependencies, including burstiness of links, their quasi-periodicity (e.g. day-night or weekly rhythms) and temporal correlation.

There exist temporal network models that mimic real-world networks in terms of certain topological features such as the number of links, clustering coefficient, degree distribution, connected components or motifs [11], [12], [13]. A system theoretical approach towards emulating temporal graphs is presented in [14]. Various approaches to human mobility are discussed in [15], [16], [17], [18], [19]. However, most existing models do not provide precise knowledge about the underlying process that generates temporal graphs. If the underlying evolutionary process is stochastic, then an observed graph sequence $G_0, \dots, G_{K-1}$ represents just one possible realization of this process. Therefore, replicating such sequences without capturing the underlying process may result in overfitting and inaccurate predictions of future network dynamics and propagating processes.

This paper aims a deeper understanding of network evolutionary processes. Specifically, we examine the *Random Walkers Induced temporal Graph (RWIG)* model [19] that generates temporal graphs based on the underlying Markov process. The essential components of RWIG are

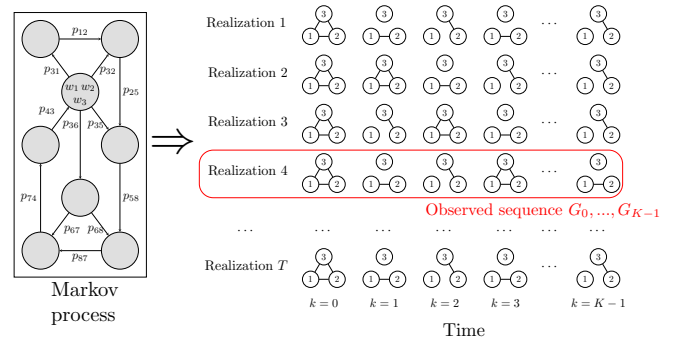


Fig. 1. Inverse problem for RWIG: non-identifiability of the initial process.

dom Walkers Induced temporal Graph (RWIG) model [19] that generates temporal graphs based on the underlying Markov process. The essential components of RWIG are

- M random walkers on a fixed underlying graph,
- the fixed underlying graph is a Markov graph on N states (nodes). The random walk on a Markov graph specifies a stochastic process that steers the walker, which we interpret as the walker policy. In principle, each independent walker w_i possesses its own Markov graph (on the same N states) specified by an $N \times N$ probability transition matrix P_i .
- the co-location principle creates the links in the contact graph G_k between walkers that are in the same state in the underlying graph at time k .

Figure 1 exemplifies $M=3$ random walkers, who traverse the same Markov graph (left) with $N=8$ states (shaded) in discrete-time steps according to the transition probabilities p_{ij} . Figure 1 (right) shows T possible contact sequences between M walkers. Initially, all walkers form a clique in the contact graph G_0 , because they are in the same state at discrete time $k=0$. At each discrete time k , RWIG generates the contact graph G_k by creating links between all walkers found in the same state in the Markov graph. Therefore, any sequence of graphs $G_0, \dots, G_{K-1}$ consists of the union of disconnected cliques, which is in complete accord with the topology of the empirical co-location contact graphs, which are released in [20]. However, RWIG cannot reproduce real-world networks with a non-clique structure.

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A physical interpretation of RWIG is a collection of individuals moving through space (e.g. city map). Each node of the underlying graph is a physical location (workplaces, homes, hospitals, schools, public transport stations, etc.) while links are physical paths between locations. RWIG assumes that the Markov process generates human motion over a set of places (states). Almasan et al. [19] derived closed-form solutions for the probability distribution of contact graphs at discrete time k . Here, we focus on the inverse problem and discuss how well RWIG can reproduce a temporal graph sequence. Specifically, our goal is to infer the initial state and the Markov graph of each walker that can generate a given K -length graph sequence $G_0, \dots, G_{K-1}$.

Solving the inverse problem for RWIG is challenging. If the evolutionary process of the temporal network is not deterministic, the observed sequence $G_0, \dots, G_{K-1}$ is only one possible realization of the stochastic process and does not replicate the entire process. An example is illustrated in Figure 1 where realization 4 of the Markov process is the only one that we observe. Hence, identifying the RWIG's parameters from a single realization may not be feasible.

Our major contributions can be summarized:

- We determine the steady-state vectors of the walkers in RWIG that generate a temporal graph sequence with the same average graph density and clique distribution as in the observed graph sequence $G_0, \dots, G_{K-1}$.
- If $G_0, \dots, G_{K-1}$ is produced by RWIG in the *steady state*, then we identify the underlying common policy matrix (the Markov graph) of the walkers that generates the observed graph sequence $G_0, \dots, G_{K-1}$.
- We demonstrate that RWIG with different policies of the walkers is able to reproduce any periodic graph sequence. We propose the algorithm that defines the Markov graph and initial state of each walker.

The paper is organized as follows. Section 2 describes RWIG and discusses the performance metrics for the inverse problem. Sections 3 and 4 discuss RWIG in the steady state. Section 3 provides the solution for the steady-state vectors of the walkers. Section 4 determines the Markov graph of the walkers. Section 5 discusses periodic graph sequences. We summarize the notation in the Appendix A.

2 RANDOM WALKERS INDUCED TEMPORAL GRAPH (RWIG)

2.1 Temporal Networks

We consider a temporal graph $G = \{G_k(\mathcal{M}, \mathcal{L}_k)\}_{k=0}^{K-1}$, consisting of a set $\mathcal{M}$ of $|\mathcal{M}|=M$ walkers connected by a set $\mathcal{L}_k$ of L_k links at discrete time $k \in \{0, 1, \dots, K-1\}$. The graph G_k is described by an $M \times M$ adjacency matrix $A[k]$ whose elements $a_{ij}[k]$ are either one or zero depending on whether there is a link between walkers w_i and w_j or not at discrete time k . For simplicity, we assume that the set of nodes is fixed and the graph is undirected without self-loops. Then all adjacency matrices $A[k] = (A[k])^T$ are real symmetric matrices. Since our focus is on the inverse problem of RWIG, we also assume that the graph G_k is composed of a set of disconnected cliques.

2.2 Description of RWIG

The random variable $X_i[k]$ denotes the state in the Markov graph of walker w_i at discrete time k and $\Pr[X_i[k] = l]$ is the probability that walker w_i is in state l in the Markov graph at discrete time k . The probability transition matrix P_i encodes the time-independent *policy* of walker w_i

$$(P_i)_{lj} = \Pr[X_i[k+1] = j | X_i[k] = l]$$

at any discrete time k . Given the initial $1 \times N$ state vector $s_i[0]$ of walker w_i (the initial position of w_i), the corresponding $1 \times N$ state vector $s_i[k]$ of walker w_i at discrete-time k is defined in [21] as

$$s_i[k] = s_i[0]P_i^k,$$

where the l -th element of the probability state vector $s_i[k]$ for walker w_i at time k is $(s_i[k])_l = \Pr[X_i[k] = l]$.

RWIG generates a sequence of contact graphs $G_0, \dots, G_{K-1}$ where the contact graph G_k consists of the union of disconnected cliques. Hence, RWIG is not able to reproduce sequences of non-clique graph structures.

Almasan et al. [19] derived the probability of an m -clique contact graph $g_k = \{\mathcal{A}_1[k], \dots, \mathcal{A}_m[k]\}$ at discrete time k

$$\Pr[G_k = g_k] = \sum_{i_1=1}^N \dots \sum_{\substack{i_m=1 \\ i_m \notin \{i_l\}_{l=1}^{m-1}}}^N \prod_{j=1}^m \prod_{w_u \in \mathcal{A}_j[k]} (s_u[k])_{i_j},$$

or, equivalently,

$$\Pr[G_k = g_k] = \sum_{\pi \in \mathcal{P}_g} \left(\prod_{\mathcal{C} \in \pi} (-1)^{|\mathcal{C}|-1} (|\mathcal{C}|-1)! \right) \prod_{\mathcal{A} \in g(\pi)} \sigma_{\mathcal{A}}[k],$$

where $\mathcal{A}_i[k]$ for all $i \in \{1, \dots, m\}$ represent the cliques formed at a discrete time k , $\mathcal{P}_g$ is the set of all partitions on g , $|\mathcal{C}|$ denotes the number of cliques $\mathcal{A}$ in cell $\mathcal{C}$ of partition π on g , $g(\pi)$ is the contact graph in which the cliques are formed by the cells $\mathcal{C}$ and $\sigma_{\mathcal{A}}[k]$ is the probability that walkers of a subset $\mathcal{A} \subseteq \mathcal{M}$ are in the same state at discrete time k

$$\sigma_{\mathcal{A}}[k] = \left(\bigodot_{w_j \in \mathcal{A}} s_j[0]P_j^k \right) u^T, \quad (1)$$

where $\odot$ denotes the Hadamard product [22] and u is the $1 \times N$ all-one vector.

Since the Markov process generates the temporal graph sequence, the probability of the observed contact graph sequence $g_0, \dots, g_{K-1}$ is given by

$$\begin{aligned} \Pr[G_0 = g_0, \dots, G_{K-1} = g_{K-1}] \\ = \Pr[G_0 = g_0] \prod_{k=1}^{K-1} \Pr[G_k = g_k | G_{k-1} = g_{k-1}], \end{aligned} \quad (2)$$

where $\Pr[G_k = g_k | G_{k-1} = g_{k-1}]$ is the conditional probability of an m -clique contact graph g_k at discrete time k given that an p -clique contact graph g_{k-1} occurs at time $k-1$ with

$$\begin{aligned} \Pr[G_k = g_k | G_{k-1} = g_{k-1}] \\ = \sum_{\substack{c_1, \dots, c_p=1 \\ c_1 \neq \dots \neq c_p}}^N \sum_{\substack{i_1, \dots, i_m=1 \\ i_1 \neq \dots \neq i_m}}^N \prod_{r=1}^p \prod_{j=1}^m \prod_{w_l \in \mathcal{A}_j[k] \cap \mathcal{A}_r[k-1]} (P_l)_{c_r, i_j}. \end{aligned} \quad (3)$$

Intuitively, equation (3) considers possible positions of M walkers in the Markov graph at time $k-1$ and k and

then evaluates the probability of walker transitions between these states. Identification of the transition probability matrix P_i and the initial state $s_i[0]$ of each walker w_i may not be analytically tractable for equation (2) because the conditional probability $\Pr[G_k=g_k|G_{k-1}=g_{k-1}]$ has $N(N-1)M$ variables and each term in (3) is the product of M variables¹. Hence, we examine another performance metric for RWIG.

Theorem 1 defines the joint probability of a clique in a contact sequence, which is generated by RWIG.

Theorem 1. Consider the RWIG model with M walkers $w_1, \dots, w_M$ where each walker w_i has the $N \times N$ transition probability matrix P_i and the initial state $s_i[0]$ with $s_i[k] = s_i[0]P_i^k$. Denote by $\mathbb{I}_C[k]$ the indicator variable that defines the existence of clique $C \subseteq \{1, \dots, M\}$ in graph G_k as

$$\mathbb{I}_C[k] = \begin{cases} 1, & \text{if a clique } C \text{ among walkers exists in } G_k, \\ 0, & \text{otherwise.} \end{cases}$$

- 1) The joint probability of a link (i, j) at discrete time $k+\kappa$ and k between walkers w_i and w_j is

$$\Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1] = s_i[k] \text{diag}(P_i^k (P_j^\kappa)^T) s_j^T[k].$$

- 2) The joint probability that a clique $C \subseteq \{1, \dots, M\}$ occurs at discrete time $k+\kappa$ and k is

$$\Pr[\mathbb{I}_C[k+\kappa]=\mathbb{I}_C[k]=1] = \left(\bigodot_{w_i \in C} s_i[k] \right) \left(\bigodot_{w_i \in C} P_i^\kappa \right) u^T.$$

- 3) The joint probability that a clique $C \subseteq \{1, \dots, M\}$ occurs from discrete time k to discrete time $k+\kappa$ is

$$\Pr[\mathbb{I}_C[k]=1, k \leq l \leq k+\kappa] = \left(\bigodot_{w_i \in C} s_i[k] \right) \left(\bigodot_{w_i \in C} P_i \right)^\kappa u^T.$$

The proof of Theorem 1 is provided in Appendix B.

2.3 Performance evaluation of RWIG

Suppose that RWIG generates the sequence $\hat{A} = (\hat{A}[0], \dots, \hat{A}[K-1])$ of adjacency matrices over K time slots. RWIG's performance can be assessed by measuring the *mean squared error (MSE)* between links in A and $\hat{A}$

$$MSE_{link}(A, \hat{A}) = \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M (a_{ij}[k] - \hat{a}_{ij}[k])^2, \quad (4)$$

where $a_{ij}[k]$ and $\hat{a}_{ij}[k]$ are real and estimated entries corresponding to the link between w_i and w_j in graph G_k .

We evaluate the MSE for RWIG. Since all random walkers move independently of each other in the Markov graph, the probability of a contact $\hat{a}_{ij}[k]$ between two walkers i and j at discrete time k depends on the corresponding probability state vectors $s_i[k]$ and $s_j[k]$

$$\begin{aligned} \hat{a}_{ij}[k] &= \sum_{l=1}^N \Pr[X_i[k]=l, X_j[k]=l] = \sum_{l=1}^N \prod_{u \in \{i, j\}} \Pr[X_u[k]=l] \\ &= s_i[k] \cdot s_j^T[k] = s_i[0] P_i^k (P_j^k)^T s_j^T[0]. \end{aligned} \quad (5)$$

Equation (5) imposes constraints on vectors $s_1[k], \dots, s_M[k]$ of the walkers at discrete time k to reproduce graph G_k exactly.

1. Each walker w_i has the $N \times N$ transition probability matrix P_i with $N(N-1)$ variables. The total number of walkers is M . The term $\prod_{j=1}^m \prod_{r=1}^p \prod_{w_l \in \mathcal{A}_{i_j} \cap \mathcal{B}_{c_r}} (P_l)_{i_j c_r}$ in (3) has polynomial degree M , because it contains one element from each matrix $P_1, \dots, P_M$.

Lemma 1. RWIG reproduces the sequence $\hat{A} = (\hat{A}[0], \dots, \hat{A}[K-1])$ of adjacency matrices with $MSE_{link}(A, \hat{A})=0$ if and only if for any discrete time k

- 1) $s_i[k]=s_j[k]=e_l$ for $a_{ij}[k]=1$ where e_l is the l -th standard basis vector, $l \in \{1, \dots, N\}$;
- 2) $s_i[k] \perp s_j[k]$ for $a_{ij}[k]=0$.

Substituting (5) into (4) yields

$$MSE_{link}(A, \hat{A}) = \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i < j} \left(a_{ij}[k] - s_i[k] s_j^T[k] \right)^2. \quad (6)$$

Given the sequence $A = (A[0], \dots, A[K-1])$, we need to identify the initial state vectors $s_{\mathcal{M}}[0] = (s_1[0], \dots, s_M[0])$ of M walkers and their Markov matrices $P_{\mathcal{M}} = (P_1, \dots, P_M)$ that minimize (6). However, identifying $s_{\mathcal{M}}[0]$ and $P_{\mathcal{M}}$ is not analytically tractable because $s_i[k]$ and $s_j[k]$ in (6) involve powers of unknown stochastic matrices $P_1, \dots, P_M$ as well as unknown initial state vectors $s_1[0], \dots, s_M[0]$. For instance, the term $s_i[k] s_j^T[k] = s_i[0] P_i^k (P_j^k)^T s_j^T[0]$ is a polynomial of degree $2(k+1)$ with $2(N^2-1)$ variables².

Equation (4) compares only the links in A and $\hat{A}$. Alternatively, we can extend $MSE_{link}(A, \hat{A})$ and compare higher-order structures, such as cliques of size r . Denote by $\mathcal{M}_r$ a set of $\binom{M}{r}$ possible combinations of r walkers from a set $\mathcal{M}$. The MSE_{clique} measure compares all cliques of size r ($2 \leq r \leq M$) in sequences A and $\hat{A}$

$$MSE_{clique}(A, \hat{A}) = \sum_{r=2}^M \frac{\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} (\mathbb{I}_C[k] - \sigma_C[k])^2}{K \binom{M}{r}}, \quad (7)$$

where $\sigma_C[k]$ is defined in (1) as the probability of clique C at discrete time k , $\binom{M}{r}$ is the number of cliques of size r in a complete graph and $\mathbb{I}_C[k]$ defines the existence of clique C in graph G_k . We analyze MSE_{link} and MSE_{clique} of RWIG in the steady state in Section 3.

3 GENERATING GRAPHS IN THE STEADY STATE

3.1 RWIG with a single stochastic matrix P in the steady state

Suppose that all walkers have the same $N \times N$ Markov transition matrix P (i.e., $P_i = P$ for all $i \in \{1, \dots, M\}$), which admits a $1 \times N$ steady-state vector $\tilde{s} = \tilde{s}P$. Then the steady-state probability vector of each walker $w_i \in \mathcal{M}$ is

$$\lim_{k \rightarrow \infty} s_i[k] = \tilde{s}.$$

with $\tilde{s}u = 1$ where u is the $N \times 1$ all-one vector [21].

Assume that all M walkers start in the same steady state $\tilde{s}$. Then $s_1[k] = \dots = s_M[k] = \tilde{s}$ and the probability of contact $\hat{a}_{ij}[k]$ between walkers w_i and w_j at discrete time k is

$$\hat{a}_{ij}[k] = s_i[k] \cdot s_j^T[k] = \tilde{s} \cdot \tilde{s}^T = \sum_{l=1}^N \tilde{s}_l^2 = p. \quad (8)$$

Applying the Cauchy-Schwarz inequality [22] to the $N \times 1$ vector $\tilde{s}$ and the $N \times 1$ all-one vector u gives

$$\left(\sum_{l=1}^N \tilde{s}_l \right)^2 \leq \sum_{l=1}^N \tilde{s}_l^2 \sum_{l=1}^N u_l^2,$$

2. Walker w_i has $N-1$ variables for the initial state vector $s_i[0]$ and N^2-N variables for the $N \times N$ transition probability matrix P_i . The total number of variables for each walker is N^2-1 . Each entry of P_i^k has polynomial degree k and $s_i[0]P_i^k$ has degree $k+1$.

or, equivalently,

$$\frac{1}{N} \leq \sum_{l=1}^N \tilde{s}_l^2 = p,$$

because $\sum_{l=1}^N \tilde{s}_l = 1$ and $0 \leq \tilde{s}_l \leq 1$. Since $\sum_{l=1}^N \tilde{s}_l^2 \leq \sum_{l=1}^N \tilde{s}_l$, we conclude that $p \in [\frac{1}{N}, 1]$. Hence, there is always a non-zero probability that walkers w_i and w_j form a contact between each other in the steady-state.

Relation (8) shows some limitations of RWIG in the steady-state for temporal network generation. First, the probability p of a contact between walkers i and j is invariant of the discrete time k , because RWIG is in the steady-state. Thus, an m -clique contact graph G is equally likely to emerge across K time slots. Second, the probability of a contact between any two walkers has a fixed probability $p = \tilde{s} \cdot \tilde{s}^T$, which is not realistic for many real networks. Third, relation (3.1) shows that the lower bound of a contact probability p is rather high for small N (e.g. $p \geq \frac{1}{4}$ for $N=4$). Hence, it is unlikely for RWIG in the steady-state to accurately reproduce any specific labeled contact sequence. Nevertheless, although all links have the same probability p to appear, the simplified RWIG model is different from any existing classical random graph model for static networks. Indeed, RWIG always produces a set of cliques. Consequently, its simplified version can also be viewed as a new random graph model for fixed graphs. Lemma 2 defines the properties of RWIG in the steady state.

Lemma 2. Consider RWIG where all M walkers have the same Markov matrix P with N states, which admits an $1 \times N$ steady-state vector $\tilde{s} = \tilde{s}P$, and $s_i[0] = \tilde{s}$ for any $i \in \{1, \dots, M\}$. Then,

- 1) the expected number of links $\bar{L}_k$ in graph G_k at any discrete time $k \in \{1, 2, \dots, K\}$ is, with $p = \sum_{l=1}^N \tilde{s}_l^2 \in [\frac{1}{N}, 1]$,

$$\bar{L}_k = \frac{M(M-1)}{2} p \in \left[\frac{M(M-1)}{2N}, \frac{M(M-1)}{2} \right];$$

- 2) the joint probability of a link (i, j) at discrete time $k + \kappa$ and k in the steady-state is

$$\Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1] = \tilde{s} \text{diag}(P^\kappa (P^\kappa)^T) \tilde{s}^T \in \left[\frac{1}{N^2}, 1 \right].$$

The proof of Lemma 2 is provided in Appendix C.1.

3.2 Performance of RWIG in the steady state with respect to MSE_{link}

We analyze MSE_{link} of RWIG, where all M walkers start in the steady state $\tilde{s}$. Substitution of (8) into (6) gives

$$\begin{aligned} MSE_{link}(A, \hat{A}) &= \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M (a_{ij}[k] - \tilde{s} \tilde{s}^T)^2 \\ &= (\tilde{s} \tilde{s}^T - \bar{a})^2 + b, \end{aligned} \quad (9)$$

where b is a non-negative constant and $\bar{a}$ is the average density of a temporal graph with $\bar{a} = \frac{2 \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M a_{ij}[k]}{KM(M-1)}$. Appendix C provides additional information on the simplification of MSE_{link} . The global minimum of (9) for an arbitrary vector $\tilde{s}$ occurs at $\tilde{s} \tilde{s}^T = \bar{a}$. Indeed, relation (9) demonstrates the variational principle of variance $Var[X]$ [21] stating that the best least-square approximation of the random variable X is its mean $E[X]$.

Since the average density $\bar{a} \in [0, 1]$ for any arbitrary graph sequence and $\tilde{s}$ is the steady-state vector with $\tilde{s} \tilde{s}^T \in [\frac{1}{N}, 1]$, equation (29) does not necessarily have a solution for a fixed N . Theorem 2 defines the accuracy of RWIG with respect to MSE_{link} as well as the conditions on steady-state vector $\tilde{s}$.

Theorem 2. Consider RWIG where all M walkers have the same Markov matrix P with N states, which admits an $1 \times N$ steady-state vector $\tilde{s} = \tilde{s}P$, and $s_i[0] = \tilde{s}$ for any $i \in \{1, \dots, M\}$. Any sequence $A = (A[0], \dots, A[K-1])$ of $M \times M$ adjacency matrices with an average graph density $\bar{a}$ can be reproduced by RWIG with

- 1) $MSE_{link}(A, \hat{A}) = \frac{\sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M (a_{ij}[k] - \frac{1}{N})^2}{K \binom{M}{2}} > 0$ if $\bar{a} < \frac{1}{N}$. The steady-state vector is $\tilde{s} = \frac{u}{N}$.
- 2) $MSE_{link}(A, \hat{A}) = \frac{Var[A]}{K \binom{M}{2}} \geq 0$ if $\bar{a} \geq \frac{1}{N}$. The steady-state vector $\tilde{s}$ satisfies $\tilde{s} \tilde{s}^T = \bar{a}$ and has finite solutions if and only if $\bar{a} \in \{\frac{1}{N}, 1\}$.

The minimal number of Markov states N to achieve the lowest MSE_{link} is $N = \lceil 1/\bar{a} \rceil$.

The proof of Theorem 2 is provided in Appendix C.3. Theorem 2 demonstrates that RWIG in the steady-state, where $s_i[0] = \tilde{s}$ for any $i \in \{1, \dots, M\}$, is able to reproduce the sequence of graphs $G_0, \dots, G_{K-1}$ accurately ($MSE_{link}(A, \hat{A}) = 0$) if and only if $Var[A] = 0$ and, consequently, $G_0, \dots, G_{K-1}$ is a sequence of complete graphs or a sequence of null graphs (there are no links between walkers). Moreover, Theorem 2 shows that for $\bar{a} \notin \{\frac{1}{N}, 1\}$, it is impossible to identify the initial steady-state vector using MSE_{link} because equation $\tilde{s} \tilde{s}^T = \bar{a}$ has infinitely many solutions. In other words, we define a class of the steady-state vectors $\tilde{s}$ that generate a temporal graph sequence with the same average graph density $\bar{a}$ as in $G_0, \dots, G_{K-1}$.

3.3 Performance of RWIG in the steady state with respect to MSE_{clique}

Sections 3.2 demonstrates that, in most cases, the initial steady-state vector $\tilde{s}$ cannot be recovered based on the MSE_{link} criterion. In this section, we show that the initial steady-state vector of the walkers can be uniquely defined using the MSE_{clique} criterion defined in (7).

We analyze MSE_{clique} of RWIG, where M walkers start in the steady-state $\tilde{s}$ and follow the same policy P . Assume that all components of the steady-state vector $\tilde{s}$ are not zero. From (1), the probability that r walkers of set $\mathcal{A} = \{w_{i_1}, \dots, w_{i_r}\}$ form a clique at discrete time k becomes

$$\sigma_{\mathcal{A}}[k] = \left(\bigodot_{w_j \in \mathcal{A}} s_j[0] P_j^k \right) u^T = \left(\bigodot_{w_j \in \mathcal{A}} \tilde{s} \right) u^T = \sum_{i=1}^N \tilde{s}_i^r, \quad (10)$$

Equation (10) is invariant to the discrete time k and provides the same value for any set of r walkers, because they all have the same steady-state vector $\tilde{s}$. Hence, for simplicity, we denote $\sigma_{\mathcal{A}}[k]$ by σ_r . Substitution of (10) into (7) gives

$$\begin{aligned} MSE_{clique}(A, \hat{A}) &= \sum_{r=2}^M \frac{\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} (\mathbb{I}_C[k] - \sigma_C[k])^2}{K \binom{M}{r}} \\ &= \sum_{r=2}^M (\sigma_r - q_r)^2 + \sum_{r=2}^M \frac{b_r}{K \binom{M}{r}}, \end{aligned} \quad (11)$$

where $b_r \geq 0$ is a constant, $q_r = \frac{\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k]}{K \binom{M}{r}}$ denotes the probability of cliques of size r in the observed graph sequence $G_0, \dots, G_{K-1}$. Appendix C provides additional information on the simplification of MSE_{clique} .

The global minimum of (11) for an arbitrary vector $\tilde{s}$ occurs when $\sigma_r = q_r$ for any $2 \leq r \leq M$. Explicitly, by (10), the global minimum of (11) must satisfy

$$\begin{cases} \sum_{i=1}^N (\tilde{s})_i = 1, \\ \sum_{i=1}^N (\tilde{s})_i^2 = q_2, \\ \dots \\ \sum_{i=1}^N (\tilde{s})_i^M = q_M. \end{cases} \quad (12)$$

The set of equations in (12) can be solved using the Newton identities for polynomials [21], [23]. Consider a polynomial $p_N(z)$ in N variables

$$p_N(z) = \sum_{r=0}^N a_r z^r = a_N \prod_{r=1}^N (z - (\tilde{s})_r), \quad (13)$$

where $a_0, \dots, a_N$ are the coefficients of a polynomial $p_N(z)$ with $a_N = 1$ and $(\tilde{s})_1, \dots, (\tilde{s})_N$ are the roots of $p_N(z)$. For each integer $r \geq 1$, the r th power sum σ_r is

$$\sigma_r = \sum_{i=1}^N \tilde{s}_i^r.$$

The relation between the coefficients $a_0, \dots, a_N$ and the power sums $\sigma_1, \dots, \sigma_N$ is derived in [21] as

$$a_r = -\frac{1}{N-r} \sum_{l=r+1}^N a_l \sigma_l. \quad (14)$$

Lemma 3 shows that RWIG in the steady state, where walkers start from $\tilde{s}$, is able to generate a temporal graph with the same probability of cliques as in $G_0, \dots, G_{K-1}$.

Lemma 3. *Let q_r be the empirical probability of cliques of size r with $1 \leq r \leq M$. The minimum of $MSE_{clique}(A, \hat{A})$ satisfies (12) and occurs when the components of the $1 \times N$ vector $\tilde{s}$ are the non-zero roots of a polynomial of order M*

$$p_M(z) = \sum_{r=0}^M a_r z^r, \quad (15)$$

where $a_r = -\frac{1}{M-r} \sum_{l=r+1}^M a_l \sigma_l$ with $a_M = 1$. The minimum number of states in the Markov graph is

$$N = M - \min\{r | a_r \neq 0 \text{ and } a_l = 0 \text{ for all } l < r\}.$$

The set of equations (12) can be transformed into (15). Given the assumption that the observed contact sequence $G_0, \dots, G_{K-1}$ between M walkers is produced by RWIG in the steady state with $N \leq M$ states, the r th power sum σ_r can be estimated from $G_0, \dots, G_{K-1}$ for any $1 \leq r \leq M$ as $\sigma_r = q_r$ for $r > 1$ and $\sigma_1 = 1$. Given the power sums $\sigma_1, \dots, \sigma_M$, we can evaluate the coefficients $a_0, \dots, a_M$ using (14). The number of states N in the Markov graph can be defined based on $a_0, \dots, a_M$, because the number of zero coefficients equals the number of zero roots of the polynomial $p_M(z)$. The non-zero roots of the polynomial in (15) provides³ the values of the steady-state vector components $(\tilde{s})_1, \dots, (\tilde{s})_N$. Any permutation of $(\tilde{s})_1, \dots, (\tilde{s})_N$ forms a steady-state vector $\tilde{s}$.

3. For polynomials of degree $N=5$ or higher, we can obtain the roots using the companion matrix in [22].

However, if $G_0, \dots, G_{K-1}$ is generated by RWIG with $N > M$ states, we cannot identify the steady-state vector $\tilde{s}$, because the probabilities $q_{M+1}, \dots, q_N$ are not observed in $G_0, \dots, G_{K-1}$ and, consequently, coefficients $a_{M+1}, \dots, a_N$ of the N -order polynomial $p_N(z)$ cannot be defined.

Lemma 3 requires that the empirical contact probabilities $q_1, \dots, q_M$ are equal to the steady-state contact probabilities $\sigma_1, \dots, \sigma_M$. The recoverability of the initial steady-state vector $\tilde{s}$ from a single realization of RWIG is discussed in Appendix D.

3.4 MSE_{link} of RWIG with different stochastic matrices $P_1, \dots, P_M$ in the steady state

Suppose that each random walker w_i has an $N \times N$ stochastic matrix P_i , which admits a steady-state distribution $\tilde{s}_i = \tilde{s}_i P_i$. Assume that the initial state $s_i[0]$ of walker w_i is equal to the steady-state probability vector $\tilde{s}_i$, i.e. $s_i[0] = \tilde{s}_i$. The probability of contact $\hat{a}_{ij}[k]$ between two walkers w_i and w_j at discrete time k is

$$\hat{a}_{ij}[k] = s_i[k] \cdot s_j^T[k] = \tilde{s}_i \tilde{s}_j^T = p_{ij} \in [0, 1]. \quad (16)$$

Relation (16) demonstrates that the probability of a contact between any two walkers at discrete time k is invariant to time in the steady state. Hence, any given m -clique contact graph G is equally likely to emerge across K time slots. This version of RWIG is more flexible than the simplified RWIG model ($P_i = P$) from Section 3.1, because it incorporates varying probabilities of contacts between walkers. RWIG in the steady state can be also viewed as another random graph model for static graphs that generates a set of cliques from steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$.

Substitution of (16) into (6) gives

$$\begin{aligned} MSE_{link}(A, \hat{A}) &= \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M \left(a_{ij}[k] - \tilde{s}_i \tilde{s}_j^T \right)^2 \\ &= \frac{1}{\binom{M}{2}} \sum_{i=1}^M \sum_{j=i+1}^M \left(\tilde{s}_i \tilde{s}_j^T - \bar{a}_{ij} \right)^2 + b, \end{aligned} \quad (17)$$

where b is a non-negative constant and $\bar{a}_{ij}$ is the average number of links between w_i and w_j in $G_0, \dots, G_{K-1}$ with $\bar{a}_{ij} = \frac{\sum_{k=0}^{K-1} a_{ij}[k]}{K}$. Appendix C provides additional information on the simplification of MSE_{link} .

Relation (17) shows that the global minimum of MSE_{link} occurs when $\tilde{s}_i \tilde{s}_j^T = \bar{a}_{ij}$ for $\forall i \neq j$. Therefore, we need to examine whether the system of equations

$$\begin{cases} \tilde{s}_i \tilde{s}_j^T = \bar{a}_{ij}, \quad \forall i, j \in \{1, \dots, M\}, \\ \tilde{s}_i u = 1, \quad \forall i \in \{1, \dots, M\}, \\ (\tilde{s})_l \geq 0, \quad \forall i \in \{1, \dots, M\}, \quad \forall l \in \{1, \dots, N\} \end{cases} \quad (18)$$

has a solution. The solution set of (18) depends on the average number of links $\bar{a}_{ij}$. For instance, if $\bar{a}_{ij} = 0$ for any two walkers w_i and w_j (the graph sequence has no links), then all walkers traverse different states of the Markov graph at any discrete time k . One possible solution for the walker w_i is the $M \times 1$ steady-state vector $\tilde{s}_i = \delta_{il}$ where δ is the Kronecker delta ($\delta_{il} = 1$ if $l=i$, and 0 otherwise). Another example, if $\bar{a}_{ij} = 1$ for any two walkers i and j (walkers form a complete graph at any discrete time k), the solution of (18) is the 1×1 steady-state vector $\tilde{s}_i = 1$ for any $i \in \{1, \dots, M\}$.

while the Markov graph has only $N=1$ state. However, if $\bar{a}_{12}=1, \bar{a}_{13}=1$ and $\bar{a}_{23}=0$, the solution of (18) does not exist⁴. Lemma 4 provides an example of static graph sequences for which system (18) always has a solution. The proof of Lemma 4 is provided in Appendix E.1.

Lemma 4. Consider RWIG where M walkers have $N \times N$ Markov transition matrices $P_1, \dots, P_M$ with an arbitrary number N of states, which admit $1 \times N$ steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$ and $s_i[0] = \tilde{s}_i$ for any $i \in \{1, \dots, M\}$. Then any sequence of graphs $G_0, \dots, G_{K-1}$ that do not change over time, where G_0 consists of the union of m disconnected cliques, can be accurately modelled by RWIG in the steady state where $N \geq m$.

The system (18) can be represented by an $M \times M$ Gram matrix $\mathcal{G}$ of all possible inner products of steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$

$$\mathcal{G} = \begin{bmatrix} g_{11} & \bar{a}_{12} & \cdots & \bar{a}_{1M} \\ \bar{a}_{12} & g_{22} & \cdots & \bar{a}_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1M} & \bar{a}_{2M} & \cdots & g_{MM} \end{bmatrix} = SS^T,$$

where $S = [\tilde{s}_1 \ \tilde{s}_2 \ \dots \ \tilde{s}_M]^T$ is an $M \times N$ matrix of the steady-state vectors and $g_{ii} = \tilde{s}_i \tilde{s}_i^T$. Geometrically, S represents the constellation of possible latent positions of M vectors on an $(N-1)$ -dimensional simplex. Unfortunately, given $\mathcal{G}$, it is impossible to recover S . If R is an $N \times N$ orthogonal matrix ($R^T R = R R^T = I$), then an $M \times N$ matrix $\tilde{S} = SR$ has the same Gram matrix⁵ [22]. Hence, the solution of (18) is identifiable only up to an orthogonal transformation⁶.

The diagonal element $g_{ii} = \tilde{s}_i \tilde{s}_i^T$ of the Gram matrix $\mathcal{G}$ can be interpreted as the probability that walker w_i remains at the same state of the Markov graph in the next time slot. Unfortunately, $g_{11}, \dots, g_{MM}$ are not known as we only observe contacts between walkers and we do not have access to the underlying Markov graph. However, the diagonal elements of $\mathcal{G}$ should satisfy certain conditions in order for system (18) to have at least one solution.

Theorem 3. Let $\mathcal{G}$ be an $M \times M$ symmetric matrix with $g_{ij} \in [0, 1]$ for any $i \neq j$ and $\mathcal{G} = U \Lambda U^T$ be the eigenvalue decomposition of $\mathcal{G}$ where $U = [u_1 \ u_2 \ \dots \ u_M]$ is an $M \times M$ orthogonal matrix formed by the scaled, real eigenvalues u_k belonging to eigenvalue $\lambda_k(\mathcal{G})$, $\Lambda = \text{diag}(\lambda_k(\mathcal{G}))$ is an $M \times M$ diagonal matrix of eigenvalues and $\lambda_1(\mathcal{G}) \geq \dots \geq \lambda_M(\mathcal{G})$. Then $\mathcal{G}$ is the Gram matrix of $1 \times N$ steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$ if and only if

- 1) $\mathcal{G}$ is positive semidefinite;
- 2) $g_{ii} \in [\frac{1}{N}, 1]$;
- 3) $N > \text{rank}(\tilde{X})$, where $\tilde{X}$ is an $(M-1) \times M$ matrix with $\tilde{X} = [x_2 - x_1 \ \dots \ x_M - x_1]^T$ and $X = U \Lambda^{1/2} = [x_1 \ \dots \ x_M]^T$;
- 4) there exists an $N \times 1$ non-zero vector n such that

$$\begin{cases} \sum_{j=1}^{\text{rank}(\mathcal{G})} n_j (\tilde{X})_{ij} = 0, \forall i \in \{1, \dots, M-1\}, \\ \frac{|\sum_{j=1}^{\text{rank}(\mathcal{G})} n_j (X)_{1j}|}{\|n\|} = \frac{1}{\sqrt{N}}. \end{cases}$$

- 5) there exists an $N \times N$ orthogonal matrix R such that $XR = [\tilde{s}_1 \ \tilde{s}_2 \ \dots \ \tilde{s}_M]$ and $\frac{1}{\sqrt{N}\|n\|} Rn = \frac{u}{N}$.

4. By Lemma 1, $\bar{a}_{12} = 1$ and $\bar{a}_{13} = 1$ implies $\tilde{s}_1 = \tilde{s}_2 = \tilde{s}_3 = (1, 0, 0, \dots)$, which in turn contradicts with $\bar{a}_{23} = 0$.

5. $\tilde{S} \tilde{S}^T = (SR)(SR)^T = S(RR^T)S^T = SS^T = \mathcal{G}$.

6. For instance, any permutation of columns of S (=steady-state vector components) produces the same matrix $\mathcal{G}$.

The proof of Theorem 3 is provided in Appendix E. Theorem 3 defines the conditions on the diagonal of $\mathcal{G}$ and defines a class of the steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$ that generate a temporal graph sequence with the same probability of a contact between any pair of walkers as in $G_0, \dots, G_{K-1}$. In addition, we provide the solution of (18) for $M = 2$ and $M = 3$ (for $N = 2$) in Appendix E.3.

4 RECOVERING THE MARKOV GRAPH OF RWIG

4.1 Recovering the stochastic matrix P_i from the steady-state vector $\tilde{s}_i$

Section 3 discusses how to find the $1 \times N$ steady-state vector $\tilde{s}_i$ of each walker w_i given the temporal graph sequence $G_0, \dots, G_{K-1}$ and the initial condition $s_i[0] = \tilde{s}_i$. To recover an $N \times N$ stochastic matrix P_i from a given steady-state vector $\tilde{s}_i$, we need to utilize properties of stochastic matrices [21]. First, P_i is an $N \times N$ square matrix of nonnegative real numbers with each row summing to 1. Second, the steady-state vector $\tilde{s}_i$ is the left eigenvector of P_i that corresponds to the eigenvalue 1. Hence, the stochastic matrix P_i can be obtained from the set of linear equations

$$\begin{cases} P_i u^T = u^T, \\ \tilde{s}_i P_i = \tilde{s}_i, \\ (P_i)_{jk} \geq 0 \quad \forall j, k \in \{1, \dots, N\}. \end{cases} \quad (19)$$

The solution of (19) is always not unique. For instance, $P_i = I$ and $P_i = [\tilde{s}_i \ \tilde{s}_i \ \dots \ \tilde{s}_i]^T$ are solutions of the system (19) for any steady-state vector $\tilde{s}_i$. Thus, the set of linear equations in (19) defines the class of stochastic matrices P_i having the same steady-state vector $\tilde{s}_i$.

Example 1. Let walker w_1 be in the steady state $\tilde{s}_1 = [\frac{1}{4} \ \frac{3}{4}]$. Then the 2×2 stochastic matrix P_1 of walker w_1 is

$$P_1 = \begin{bmatrix} -2 + 3p_{22} & 3 - 3p_{22} \\ 1 - p_{22} & p_{22} \end{bmatrix},$$

where $\frac{2}{3} \leq p_{22} \leq 1$.

As a remark, a set of linear equations in (19) are sufficient to define the stochastic matrix P_i if and only if $s_i[0] = \tilde{s}_i$, because it is possible that $\lim_{k \rightarrow \infty} s_i[k] = \lim_{k \rightarrow \infty} s_i[0] P_i^k \neq \tilde{s}_i$ for some $s_i[0]$ and P_i from system (19). However, if the Markov chain is irreducible, aperiodic, and positive recurrent [21], then any initial vector $s_i[0]$ converges to the unique steady-state vector $\tilde{s}_i$.

Example 2. Let walker w_1 be in the steady state $\tilde{s}_1 = [\frac{1}{2} \ \frac{1}{2}]^T$. Then the 2×2 stochastic matrix P_1 of walker w_1 is

$$P_1 = \begin{bmatrix} p_{22} & 1 - p_{22} \\ 1 - p_{22} & p_{22} \end{bmatrix},$$

where $0 \leq p_{22} \leq 1$. Suppose $p_{22} = 0$ and $P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. If $s_1[0] = [\frac{1}{2} \ \frac{1}{2}]^T$ then $\tilde{s}_1 = \lim_{k \rightarrow \infty} s_i[0] P_i^k = [\frac{1}{2} \ \frac{1}{2}]^T$. However, if $s_1[0] = [1 \ 0]^T$, the steady-state vector $\tilde{s}_1$ does not exist.

4.2 Recovering the stochastic matrix P from the steady-state contact sequence

Assume that the contact sequence $G_0, \dots, G_{K-1}$ is generated by RWIG in the steady state where all walkers have the same Markov policy P and steady-state vector $\tilde{s}$. We demonstrate that, under this assumption, we can recover the Markov

graph using the time correlation between cliques. Since $s_i[k]=\tilde{s}$ for any walker w_i in the steady state, we rewrite the joint probability that a clique $C \subseteq \{1, \dots, M\}$ occurs at discrete time k and $k+1$ from Theorem 1 as

$$\Pr[\mathbb{I}_C[k]=1, \mathbb{I}_C[k+1]=1] = \left(\bigodot_{w_i \in C} \tilde{s} \right) \left(\bigodot_{w_i \in C} P \right) u^T$$

$$= \sum_{c_0=1}^N \sum_{c_1=1}^N (\tilde{s}_{c_0} P_{c_0 c_1})^{|C|}. \quad (20)$$

Equation (20) is independent of the choice of the walker set C and the discrete time k and is determined solely by the size of C . For simplicity, we denote the joint probability in (20) by $\sigma_{r,1}$ where $r = |C|$.

Denote by W the $N \times N$ matrix obtained from matrix P and steady-state vector $\tilde{s}$

$$W = \begin{bmatrix} \tilde{s}_1 P_{11} & \tilde{s}_1 P_{12} & \cdots & \tilde{s}_1 P_{1N} \\ \tilde{s}_2 P_{21} & \tilde{s}_2 P_{22} & \cdots & \tilde{s}_2 P_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{s}_N P_{N1} & \tilde{s}_N P_{N2} & \cdots & \tilde{s}_N P_{NN} \end{bmatrix} = P \odot \tilde{s}^T.$$

The sum of elements in i -th row of matrix W is $\sum_{j=1}^N \tilde{s}_i P_{ij} = \tilde{s}_i$. The sum of elements in j -th column of matrix W is $\sum_{i=1}^N \tilde{s}_i P_{ij} = \tilde{s}_j$, because $\tilde{s}$ is the steady-state vector of P with $\tilde{s}P = \tilde{s}$. Thus, $Wu^T = \tilde{s}^T$ and $uW = \tilde{s}$. The sum of all elements of W is $\sum_{i=1}^N \sum_{j=1}^N w_{ij} = uWu^T = 1$. Furthermore, for each integer $r > 1$ the r th power sum of elements of W defines the joint probability that graphs G_k and G_{k+1} contain the same clique of size r

$$\sum_{i=1}^N \sum_{j=1}^N w_{ij}^r = \sigma_{r,1}.$$

Theorem 4 shows that the elements of W can be identified using the Newton identities for polynomials [21], [23]. The proof of Theorem 4 is provided in Appendix F.

Theorem 4. Consider the $N \times N$ matrix $W = P \odot \tilde{s}^T$. Given the steady-state vector $\tilde{s}$ and the steady-state joint probability $\sigma_{r,1}$ for each integer $2 \leq r \leq N^2$, the elements of matrix W are the roots of a polynomial of order N^2

$$p_{N^2}(z) = \sum_{r=0}^{N^2} a_r z^r, \quad (21)$$

where $a_r = -\frac{1}{N^2-r} \sum_{l=r+1}^{N^2} a_l \sigma_{l,1}$ with $a_{N^2} = 1$. The position of the r th root z_r of (21) in the matrix W is defined by the solution of the placement problem

$$\begin{cases} \sum_{r=1}^{N^2} y_{ir} z_r = \tilde{s}_i, & \text{for all } i \in \{1, \dots, N\}, \\ \sum_{r=1}^{N^2} y_{rj} z_r = \tilde{s}_j, & \text{for all } j \in \{1, \dots, N\}, \\ \sum_{j=1}^N y_{rj} = 1, & \text{for all } r \in \{1, \dots, N^2\}, \\ \sum_{i=1}^N y_{ir} = 1, & \text{for all } r \in \{1, \dots, N^2\}, \\ \sum_{r=1}^{N^2} y_{ir} = N, & \text{for all } i \in \{1, \dots, N\}, \\ \sum_{r=1}^{N^2} y_{rj} = N, & \text{for all } j \in \{1, \dots, N\} \end{cases} \quad (22)$$

where $y_{ir}, y_{rj} \in \{0, 1\}$ for all $i, j \in \{1, \dots, N\}$ are the binary variables that define if the r th root z_r , $r \in \{1, \dots, N^2\}$ has position w_{ij} in matrix W .

Given the assumption that $G_0, \dots, G_{K-1}$ is produced by RWIG in the steady state, we identify the steady-state vector

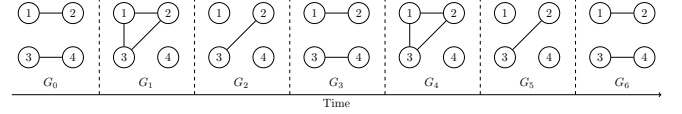


Fig. 2. 3-periodic sequence for a temporal graph with 4 walkers.

$\tilde{s}$ using Lemma 3 and evaluate the joint probability $\sigma_{r,1}$ for any $1 < r \leq M^2$. We define the coefficients $a_0, \dots, a_{N^2}$ and identify the roots $\{z_r\}_{r=1}^{N^2}$ of (21). The solution of (22) is not unique because W and W^T satisfy $Wu^T = \tilde{s}^T$ and $uW = \tilde{s}$. The set of stochastic matrices P is derived from $\tilde{s}$ and W .

Example 3. Consider the Markov graph with $N=3$ states and $M=9$ walkers that traverse the Markov graph according to the 3×3 Markov transition probability matrix

$$P = \begin{bmatrix} 0.567 & 0.157 & 0.276 \\ 0.373 & 0.276 & 0.351 \\ 0.327 & 0.502 & 0.171 \end{bmatrix}.$$

Given the steady-state probability q_r of the clique of size r for $r \leq 9$, Lemma 3 defines the steady-state vector $\tilde{s} = [0.447, 0.284, 0.269]$. Given the steady-state joint probability $\sigma_{r,1}$ that the same clique of size r occurs in two adjacent time slots for $r \leq 9$, the roots of (21) are $z = [0.254, 0.135, 0.123, 0.106, 0.1, 0.088, 0.078, 0.07, 0.046]$. From (22), there are only two possible arrangements or roots $\{z_r\}_{r=1}^9$ in the W matrix

$$W_1 = \begin{bmatrix} 0.254 & 0.07 & 0.123 \\ 0.106 & 0.078 & 0.1 \\ 0.088 & 0.135 & 0.046 \end{bmatrix}, \quad W_2 = W_1^T.$$

Dividing each row i of W_1 and W_2 by the corresponding component $\tilde{s}_i$ of the steady-state vector produces transition probability matrices P_1 (the initial matrix) and P_2

$$P_1 = \begin{bmatrix} 0.567 & 0.157 & 0.276 \\ 0.373 & 0.276 & 0.351 \\ 0.327 & 0.502 & 0.171 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.567 & 0.236 & 0.197 \\ 0.248 & 0.276 & 0.476 \\ 0.459 & 0.37 & 0.171 \end{bmatrix}.$$

Example 3 illustrates how the initial Markov graph can be inferred from Theorem 4. Additionally, we demonstrate in Appendix D how Theorem 4 is applied to identify the underlying topology of the real transportation system (PATH rail system) using the contact data of random walkers.

5 RWIG FOR PERIODIC SEQUENCES

Section 3 demonstrates that RWIG in the steady state can accurately reproduce only static graph sequences $G_0, \dots, G_{K-1}$ where $G_0 = \dots = G_{K-1}$. We call such graph sequences 1-periodic because $\forall k \in \{0, \dots, K-2\} G_k = G_{k+p}$ for $p=1$. In this section, we examine whether RWIG can accurately reproduce p -periodic graph sequences where $p > 1$. Our motivation for studying periodic sequences is that many real-world systems possess a quasi-periodic dynamic that repeats during a certain period of time [14], [24].

Definition 1. The graph sequence $G_0, \dots, G_{K-1}$ is called p -periodic if there is a positive integer $p < K-1$ with $G_k = G_{k+p}$ for any $k \in \{0, 1, \dots, K-p-1\}$.

An example of 3-periodic sequence for a temporal graph with $M = 4$ walkers is shown in Figure 2.

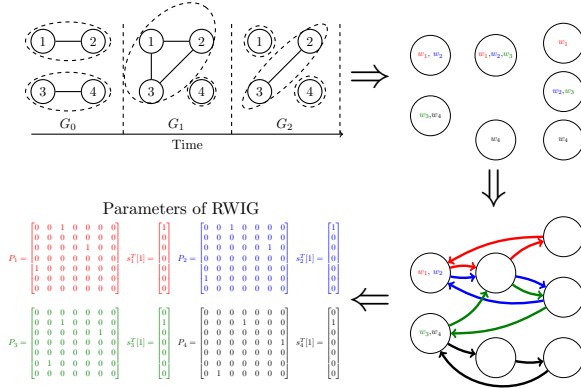


Fig. 3. RWIG with $N=7$ states for 3-periodic graph sequence from Fig.2.

Lemma 5. *The RWIG model where the walkers have the same $N \times N$ Markov transition matrix P can accurately reproduce only 1-periodic graph sequences $G_0, \dots, G_{K-1}$.*

The proof of Lemma 5 is provided in Appendix G.1.

Consider RWIG where M walkers have $N \times N$ transition matrices $P_1, \dots, P_M$. Since $a_{ij}[k] = a_{ij}[k+p]$ for any walkers w_i and w_j at any discrete time $k \in \{0, 1, \dots, K-p-1\}$, RWIG reproduces p -periodic graph sequence $G_0, G_1, \dots, G_{K-1}$ accurately if and only if $a_{ij}[k] = \hat{a}_{ij}[k] = \hat{a}_{ij}[k+p]$ for any $k \in \{0, \dots, K-p-1\}$ and any $i, j \in \{1, \dots, M\}$ where $\hat{a}_{ij}[k] = s_i[k] s_j^T[k]$ is defined in (5). We assume that each walker w_i traverses the Markov graph with period p (i.e. $s_i[k] = s_i[k+p]$ for any $k \in \{0, \dots, K-p-1\}$) and prove that any p -periodic graph sequence $G_0, \dots, G_{K-1}$ can be reproduced by RWIG if any graph G_k contains a set of cliques.

First, we demonstrate in Figure 3 the solution for the 3-periodic graph sequence from Figure 2. We introduce $N = 7$ states in the Markov graph because there are 2 cliques in G_0 , 2 cliques in G_1 and 3 cliques in G_2 . Each state of the Markov graph corresponds to one of the cliques in the temporal graph at discrete time $k \in \{0, 1, 2\}$. Initially, the walkers w_1, w_2 and w_3, w_4 are placed in the same state as they have a contact in G_0 (see Figure 3, bottom right). At each discrete time k , walkers move to the same state if they have a contact between each other. Otherwise, a walker w_i moves to the state that contains w_i in discrete time k (see Figure 3, top right). The transitions of the walkers are shown in Figure 3 (red for walker w_1 , blue for walker w_2 , green for walker w_3 and black for walker w_4). The total number of walker transitions in the Markov graph, or, equivalently, the total number of non-zero elements in the transition matrices $P_1, \dots, P_4$ is sum of walkers periods, i.e., $\sum_{j=1}^4 p_i = 12$.

Lemma 6. *Any p -periodic graph sequences $G_0, \dots, G_{K-1}$ can be accurately reproduced by RWIG where M walkers have different $N \times N$ Markov transition matrices $P_1, \dots, P_M$ and $N = \sum_{k=0}^{p-1} n_c[k]$, $n_c[k]$ is the number of cliques in G_k .*

The proof of Lemma 6 is provided in Appendix G.2.

Lemma 6 shows that any periodic graph sequence can be represented by RWIG where the Markov graph has $N = \sum_{k=0}^{p-1} n_c[k]$ states and each walker w_i follows p -periodic walk. We formulate two research questions:

- 1) What is the minimal period of each walker to reproduce p -periodic graph sequence?

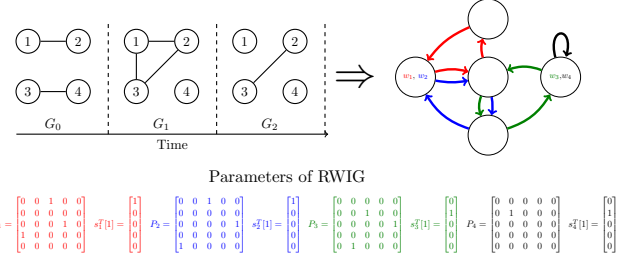


Fig. 4. RWIG with $N=5$ states for 3-periodic graph sequence from Fig.2. The transitions of the walkers are shown in red for w_1 , blue for w_2 , green for w_3 and black for w_4 .

- 2) What is the minimal number of states N in the Markov graph to reproduce p -periodic graph sequence?

Intuitively, the shorter the period p_i of each walker w_i , the fewer states are needed to describe the walk of w_i and, consequently, the fewer number of states N should be in the Markov graph. However, we demonstrate that the minimal period of the walkers does not imply the minimal number of states in the Markov graph.

Lemma 7. *Any non-zero contact sequence between walkers w_i and w_j with period p , where p is the minimal period of the sequence, can be reproduced by RWIG where walkers w_i and w_j traverse the Markov graph with periods p_i and p_j such that*

- $LCD(p_1, p_2) \bmod p = 0$, where $LCD(p_1, p_2)$ is the least common denominator of p_1 and p_2 .
- $\min(p_1, p_2) \geq \frac{LCD(p_1, p_2)}{p} \sum_{k=0}^{p-1} a_{ij}[k]$
- $\nexists k_1, k_2 < LCD(p_1, p_2)$ with $a_{ij}[k_1] = a_{ij}[k_2] = 1$ such that

$$\begin{cases} k_1 \bmod p_1 = k_2 \bmod p_1, \\ k_1 \bmod p_2 = k_2 \bmod p_2. \end{cases}$$

Lemma 7 defines the set of periods (p_i, p_j) for walkers w_i and w_j to reproduce the observed contact sequence between them. For instance, the contact sequence $[1 \ 0 \ 0]$ between walkers w_3 and w_4 can be accurately reproduced by RWIG if $(p_3, p_4) \in \{(1, 3), (2, 3), (3, 3), (3, 1), (3, 2)\}$. The proof of Lemma 7 is provided in Appendix G.3.

The minimal period p_i of walker w_i in p -periodic contact sequence is the period, which is present in all sets between walker w_i and other walkers. For instance, the 3-periodic graph sequence from Figure 2 can be modelled by RWIG where walkers w_1, w_2, w_3, w_4 have periods $p_1 = p_2 = p_3 = 3$ and $p_4 = 1$. Hence, the minimal number of states in the Markov graph is $N = 5$ because walkers w_1 and w_2 require 4 states (w_1 and w_2 should traverse three different states while two of them should be shared) and walker w_4 requires one additional state (walker w_4 , which has no contacts with w_1 and w_2 , stays in the same state). Figure 4 demonstrates the parameters of RWIG with $N=5$ states for the 3-periodic graph sequence from Figure 2. The total number of transitions in the Markov graph with $N=5$ states, or, equivalently, the total number of non-zero elements in the transition matrices $P_1, \dots, P_4$ is 10.

However, $N=5$ states is not the minimal number of states in the Markov graph. Suppose that all walkers traverse the Markov graph with period $p = 3$. We propose Algorithm 1, which is a heuristic method that constructs the Markov graph with a small number of states for p -periodic graph sequence $G_0, \dots, G_{K-1}$. First, Algorithm 1 se-

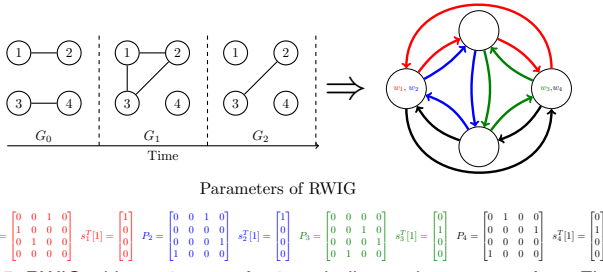


Fig. 5. RWIG with $N=4$ states for 3-periodic graph sequence from Fig.2. The transitions of the walkers are shown in red for w_1 , blue for w_2 , green for w_3 and black for w_4 .

lects walker w_1 and generates $N = p_1$ states in the Markov graph (walker w_1 should traverse p_1 different states) where the states of the walker are $X_1[k] = k$ for $k \in \{1, \dots, p_1\}$. Then Algorithm 1 chooses the next walker w_2 that traverses p_2 states. If walker w_2 has a contact with walker w_1 at discrete time $k \leq K$ (function "GetFirstContact"), then walker w_2 shares one of the existing states with w_1 , i.e., $X_2[k \bmod p_2] = X_1[k \bmod p_1]$. However, if the walker w_2 has no contact with walker w_1 when w_2 was on k -th state ($k \leq p_2$), w_2 chooses the first available state $l \in \{1, \dots, N\}$ in the Markov graph (function "GetFreeState")⁷. If such states are not available, we add an additional state to the Markov graph for walker w_2 . Thus, Algorithm 1 iteratively processes each walker w_i until all M walkers have been considered.

Algorithm 1 Markov Graph Generator (Periodic sequences)

Input: periods $p_1, \dots, p_M$ of M walkers, contact graph sequence $G_0, \dots, G_{K-1}$.

Output: periodic states sequence $X_1, \dots, X_M$ of walkers, the size N of the Markov graph.

```

1:  $N \leftarrow 0$ 
2: for  $i \leftarrow 1$  to  $M$  do
3:    $X_i \leftarrow 0_{p_i \times 1}$ 
4:   for  $p \leftarrow 1$  to  $p_i$  do
5:     for  $k \leftarrow p$  to  $K$  step  $p_i$  do
6:        $j \leftarrow \text{GETFIRSTCONTACT}(G_k, i)$ 
7:       if  $j \neq 0$  then  $X_i[p] \leftarrow X_j[k \bmod p_j]$  break
8:       end if
9:     end for
10:    if  $X_i[p] = 0$  then
11:       $s \leftarrow \text{GETFREESTATE}(N, [X_1, \dots, X_i])$ 
12:      if  $s = 0$  then
13:         $X_i[p] \leftarrow N + 1, N \leftarrow N + 1$ 
14:      else
15:         $X_i[p] \leftarrow s$ 
16:      end if
17:    end if
18:  end for
19: end for
20: return  $[X_1 \dots X_M], N$ 

```

Figure 5, obtained by Algorithm 1, shows that the 3-periodic graph sequence from Figure 2 can be modelled by RWIG with $N=4$ states. The total number of transitions in

⁷ State $l \in \{1, \dots, N\}$ is available for walker w_2 if $X_1[k] \neq l$ for any $(k \bmod p_1) = l$ (w_2 never meets w_1 at state l) and $X_2[t] \neq l$ for $t < k$ (w_2 have not visited state l).

the Markov graph with $N=4$ states is 12, which is more compared to Figure 4. Hence, Algorithm 1 demonstrates that the minimal periods of the walkers do not imply the minimal number of states in the Markov graph.

6 DISCUSSION

We examined the inverse problem for RWIG in the steady state. If all walkers have the same transition probability matrix P and start from the steady-state vector $\tilde{s}$, RWIG is able to accurately reproduce only a sequence of complete or null graphs. Any other temporal graph $G_0, \dots, G_{K-1}$ with M walkers can be approximately modelled by the graph sequence that has the same probability of a clique of size r and the joint probability $\sigma_{r,1}$ for each $r \leq M$. Furthermore, we demonstrate that inferring the initial ergodic Markov process is possible: given $G_0, \dots, G_{K-1}$, we derive an exact analytical solution that defines the transition probability matrix P of the walkers. Our findings are based on fundamental results of Newton in polynomial theory and functional analysis. The scalability and computational complexity of our methodology are driven by the $O(n^2)$ complexity of existing root-finding methods for n -order polynomials.

If walkers have different transition probability matrices $P_1, \dots, P_M$ that have different steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$, RWIG can accurately reproduce only a sequence of m -clique graphs that do not change over time. The lowest possible mean squared error (MSE) occurs when $\tilde{s}_i \tilde{s}_j^T = \bar{a}_{ij}$ for all pairs of walkers w_i and w_j , where $\bar{a}_{ij}$ is the average number of links between i and j in $G_0, \dots, G_{K-1}$. We imposed several constraints on steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$ to achieve this MSE value. However, the general solution for $\tilde{s}_1, \dots, \tilde{s}_M$ for arbitrary N and M remains unknown.

For periodic sequences, we have proven that, if the walkers follow the same policy P in the Markov graph, RWIG can accurately reproduce only 1-periodic sequences. We have also demonstrated that any p -periodic sequence can be reproduced by RWIG where M walkers traverse the Markov graph with different transition probability matrices $P_1, \dots, P_M$. We provide the lemma that defines the minimal period p_i of each walker w_i .

There are several future directions for this research. First, we have identified the constraints on the temporal graph sequence, which can be reproduced by RWIG with the lowest MSE. However, if a given temporal graph sequence $G_0, \dots, G_{K-1}$ does not satisfy these constraints, what are the parameters of RWIG in the steady state to approximate $G_0, \dots, G_{K-1}$? Second, how to infer the parameters of RWIG when the sequence $G_0, \dots, G_{K-1}$ is neither periodic nor generated in the steady state? Finally, our findings are valid only for graph sequences that can be generated by RWIG. Therefore, we emphasize the importance of solving the inverse problem for other generative models that produce non-clique structures.

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APPENDIX A SYMBOLS

Only when explicitly mentioned, will we deviate from the symbols outlined here.

General Notation

k	discrete time
u	the $1 \times N$ all-one vector
e_l	$1 \times N$ l -th standard basis vector, $l \in \{1, \dots, N\}$
$\odot$	Hadamard product
$\lceil x \rceil$	the smallest integer value greater than or equal to x .
$E[X]$	mean of the random variable X
$Var[X]$	variance of the random variable X
$p_N(z)$	symmetric polynomial in N variables
$LCD(p_1, p_2)$	the least common denominator (LCD) of numbers p_1 and p_2
$k_1 \bmod p_1$	the remainder of the division of k_1 by k_2
RWIG	
N	number of states (nodes) in the Markov graph
M	number of random walkers
$\mathcal{M}_r$	set of $\binom{M}{r}$ possible combinations of r walkers from a set $\mathcal{M}$
$\mathcal{M}$	the set of M walkers with $ \mathcal{M} = M$
w_i	label of walker w_i
P_i	$N \times N$ transition probability matrix of walker w_i
$P_{\mathcal{M}}$	set of transition probability matrices $P_1, \dots, P_M$ of M walkers
$X_i[k]$	random variable of the state of walker w_i at discrete time k
$s_i[k]$	$1 \times N$ Markov states probability distribution of walker w_i at discrete time k
$s_{\mathcal{M}}[0]$	set of initial state vectors $s_1[0], \dots, s_M[0]$ of M walkers
$\mathcal{A}_i[k] \subseteq \mathcal{M}$	clique i of walkers at discrete time k (i.e. a subset of the walker set $\mathcal{M}$)
$C \subseteq \mathcal{M}$	subset of walkers
g_k	Contact graph (i.e. partition on the set $\mathcal{M}$) realisation at discrete time k
$\mathcal{P}_g$	the set of all partitions on g
$\mathcal{C}$	cell of a partition on a contact graph (i.e. a subset of subsets of walkers)
$g(\pi)$	The contact graph associated with partition π on the contact graph g (i.e. the amassed-clique contact graph formed by coalescing cliques in g)
$\sigma_{\mathcal{A}_i[k]}$	probability that all walkers in the subset $\mathcal{A}_i[k]$ are in the same Markov state at time k (i.e. at the same node in the underlying graph)
σ_r	steady-state probability that r walkers are in the same Markov state ($=r$ th power sum of the steady-state components)
$\Pr[X_i[k]=l]$	probability that walker w_i is in state l in the Markov graph at discrete time k
p	probability of a link between walkers in the steady state with policy P
$\tilde{s}$	steady-state vector of all walkers

 $\tilde{s}_i$ p_{ij} $\mathcal{G}$ n

Temporal Graphs

 G_k or $G_k(\mathcal{M}, \mathcal{L}_k)$ $\mathcal{L}_k$ L_k $\mathcal{L}$ L $A[k]$ $a_{ij}[k]$ $\bar{a}_{ij}$ $\bar{a}$ $A=(A[0], \dots, A[K-1])$ $\hat{A}=(\hat{A}[0], \dots, \hat{A}[K-1])$ $\mathbb{I}_C[k]$ $\bar{L}_k$ q_r $\sigma_{r,1}$

steady-state vector of walker w_i
probability of a link between walkers w_i and w_j in the steady state with the different policies P_i and P_j
 $M \times M$ Gram matrix
a nonzero normal vector (or orthogonal vector) to the hyperplane containing the points corresponding to the steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$

temporal graph at discrete time k with walkers (nodes) $\mathcal{M}$ and links $\mathcal{L}_k$
set of links in G_k
number of links in G_k , $L_k = |\mathcal{L}_k|$
set of all links in G_k , $\mathcal{L} = \bigcup_{k=1}^T \mathcal{L}_k$
the total number of links in $G_0, \dots, G_{K-1}$
 $M \times M$ adjacency matrix of graph G_k
 (i, j) -element of an adjacency matrix $A[k]$
average number of links between walkers w_i and w_j over K time slots.
average number of links (average density) in graph sequence $G_0, \dots, G_{K-1}$
the observed sequence of $M \times M$ adjacency matrices over K time slots.
the sequence of $M \times M$ adjacency matrices, which is generated by RWIG over K time slots.
the indicator variable that defines the existence of clique between walkers of set C in graph G_k
expected number of links in graph G_k
the probability of cliques of size r in $G_0, \dots, G_{K-1}$
the steady-state joint probability that the same clique of size r occurs in two adjacent time slots

APPENDIX B

RWIG: JOINT AND CONDITIONAL PROBABILITY OF A LINK

Proof of Theorem 1. Consider the RWIG model where walkers have different transition probability matrices $P_1, P_2, \dots, P_M$ and initial states $s_1[0], s_2[0], \dots, s_M[0]$. The joint probability of a link (i, j) at time $k+1$ and k is

$$\begin{aligned} & \Pr[a_{ij}[k+1]=1, a_{ij}[k]=1] \\ &= \sum_{c_0=1}^N \sum_{c_1=1}^N \Pr[X_i[k+1]=c_1, X_j[k+1]=c_1, X_i[k]=c_0, X_j[k]=c_0]. \end{aligned}$$

Since walkers w_i and w_j move independently of each other in the Markov graph, then

$$\begin{aligned}
& \Pr[a_{ij}[k+1]=1, a_{ij}[k]=1] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N \prod_{u \in \{i,j\}} \Pr[X_u[k+1]=c_1, X_u[k]=c_0] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N \prod_{u \in \{i,j\}} \Pr[X_u[k+1]=c_1 | X_u[k]=c_0] \Pr[X_u[k]=c_0] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N (P_i)_{c_0 c_1} \cdot (s_i[k])_{c_0} \cdot (P_j)_{c_0 c_1} \cdot (s_j[k])_{c_0} \\
&= \sum_{c_0=1}^N \left((s_i[k])_{c_0} (s_j[k])_{c_0} \sum_{c_1=1}^N (P_i)_{c_0 c_1} (P_j)_{c_0 c_1} \right) \\
&= \sum_{c_0=1}^N \left((s_i[k])_{c_0} (s_j[k])_{c_0} \sum_{c_1=1}^N (P_i)_{c_0 c_1} (P_j^T)_{c_1 c_0} \right) \\
&= \sum_{c_0=1}^N (s_i[k])_{c_0} (s_j[k])_{c_0} (P_i P_j^T)_{c_0 c_0},
\end{aligned}$$

which we write as

$$\begin{aligned}
\Pr[a_{ij}[k+1]=1, a_{ij}[k]=1] &= s_i[k] \text{diag}(P_i P_j^T) s_j^T[k] \\
&= s_i[0] P_i^k \text{diag}(P_i P_j^T) (P_j^k)^T s_j^T[0]. \quad (23)
\end{aligned}$$

Equation (23) can be extended to compute the joint probability of a link (i, j) at time $k+\kappa$ and k

$$\begin{aligned}
& \Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N \Pr[X_i[k+\kappa]=c_1, X_j[k+\kappa]=c_1, X_i[k]=c_0, X_j[k]=c_0] \mathbb{I}_C[k] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N \prod_{u \in \{i,j\}} \Pr[X_u[k+\kappa]=c_1, X_u[k]=c_0] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N ((P_i)_{c_0 c_1}^\kappa \cdot (s_i[k])_{c_0}) \cdot ((P_j)_{c_0 c_1}^\kappa \cdot (s_j[k])_{c_0}) \\
&= \sum_{c_0=1}^N \left((s_i[k])_{c_0} (s_j[k])_{c_0} \sum_{c_1=1}^N (P_i)_{c_0 c_1}^\kappa (P_j)_{c_0 c_1}^\kappa \right),
\end{aligned}$$

which we write as

$$\Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1] = s_i[k] \text{diag}(P_i^\kappa (P_j^\kappa)^T) s_j^T[k]. \quad (24)$$

The joint probability that a link (i, j) will be present from discrete time k to discrete time $k+\kappa$ is

$$\begin{aligned}
& \Pr[a_{ij}[k]=1, \dots, a_{ij}[k+t]=1] \\
&= \sum_{c_0=1}^N \dots \sum_{c_t=1}^N \Pr[X_i[k]=c_0, X_j[k]=c_0, \dots, X_i[k+t]=c_t, X_j[k+t]=c_t] \\
&= \sum_{c_0=1}^N \dots \sum_{c_t=1}^N \prod_{u \in \{i,j\}} \Pr[X_u[k]=c_0, \dots, X_u[k+t]=c_t].
\end{aligned}$$

Since the walks of any walker w_i are generated by the Markov process, then

$$\begin{aligned}
& \Pr[X_i[k]=c_0, \dots, X_i[k+\kappa]=c_t] \\
&= \Pr[X_i[k]=c_0] \cdot \prod_{r=0}^{t-1} \Pr[X_i[k+r+1]=c_{r+1} | X_i[k+r]=c_r] \\
&= (s_i[k])_{c_0} \cdot \prod_{r=0}^{t-1} (P_i)_{c_r c_{r+1}},
\end{aligned}$$

and, consequently,

$$\begin{aligned}
& \Pr[a_{ij}[k]=1, \dots, a_{ij}[k+\kappa]=1] \\
&= \sum_{c_0=1}^N \dots \sum_{c_t=1}^N (s_i[k])_{c_0} \cdot \prod_{r=0}^{t-1} (P_i)_{c_r c_{r+1}} (s_j[k])_{c_0} \cdot \prod_{r=0}^{t-1} (P_j)_{c_r c_{r+1}} \\
&= \sum_{c_0=1}^N (s_i[k])_{c_0} (s_j[k])_{c_0} \sum_{c_1=1}^N \dots \sum_{c_t=1}^N \prod_{u \in \{i,j\}} (P_u)_{c_0 c_1} \dots (P_u)_{c_{t-1} c_t} \\
&= \sum_{c_0=1}^N \left((s_i[k])_{c_0} (s_j[k])_{c_0} \sum_{c_t=1}^N (P_i \odot P_j)_{c_0 c_t}^\kappa \right),
\end{aligned}$$

which we write as

$$\Pr[a_{ij}[k]=1, \dots, a_{ij}[k+\kappa]=1] = (s_i[k] \odot s_j[k]) (P_i \odot P_j)_{c_0 c_t}^\kappa u^T, \quad (25)$$

where u is the $1 \times N$ all-one vector and $\odot$ denotes the Hadamard product.

We extend the joint probabilities to cliques. Denote by $\mathbb{I}_C[k]$ the indicator variable that defines the existence of clique between walkers from C in graph G_k as

$$\mathbb{I}_C[k] = \begin{cases} 1, & \text{if a clique among walkers from } C \text{ exists in } G_k, \\ 0, & \text{otherwise.} \end{cases}$$

The joint probability that walkers from set $C \subseteq \{1, \dots, M\}$ have contacts between each other at time $k+1$ and k is

$$\begin{aligned}
& \Pr[\mathbb{I}_C[k+1]=\mathbb{I}_C[k]=1] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N \Pr[\forall w_u \in C X_u[k+1]=c_1, X_u[k]=c_0].
\end{aligned}$$

Independence of the walkers indicates that

$$\begin{aligned}
& \Pr[\mathbb{I}_C[k+1]=\mathbb{I}_C[k]=1] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N \prod_{w_u \in C} \Pr[X_u[k+1]=c_1, X_u[k]=c_0] \\
&= \sum_{c_0=1}^N \sum_{c_1=1}^N \prod_{w_u \in C} (P_u)_{c_0 c_1} \cdot (s_u[k])_{c_0},
\end{aligned}$$

which we write as

$$\Pr[\mathbb{I}_C[k+1]=\mathbb{I}_C[k]=1] = \left(\bigodot_{w_u \in C} s_u[k] \right) \left(\bigodot_{w_u \in C} P_u \right) u^T, \quad (26)$$

Equation (26) is the generalization of equation (23). We extend (24) and define the joint probability that walkers

from set $C \subseteq \{1, \dots, M\}$ have contacts between each other at time $k+\kappa$ and k as

$$\Pr[\mathbb{I}_C[k+\kappa]=\mathbb{I}_C[k]=1] = \left(\bigodot_{w_u \in C} s_u[k] \right) \left(\bigodot_{w_u \in C} P_u^\kappa \right) u^T. \quad (27)$$

From equation (25), the joint probability that walkers from set $C \subseteq \{1, \dots, M\}$ have contacts between each other have contacts between each other from time k to time $k+\kappa$ is

$$\Pr[\mathbb{I}_C[k]=1 \ k \leq l \leq k+\kappa] = \left(\bigodot_{w_u \in C} s_u[k] \right) \left(\bigodot_{w_u \in C} P_u \right)^\kappa u^T. \quad (28)$$

APPENDIX C

MSE_{link} AND MSE_{clique} OF RWIG IN THE STEADY STATE

C.1 Proof of Lemma 2

Lemma 2. Consider RWIG where all M walkers have the same Markov matrix P with N states, which admits an $1 \times N$ steady-state vector $\tilde{s} = \tilde{s}P$, and $s_i[0] = \tilde{s}$ for any $i \in \{1, \dots, M\}$.

- 1) The expected number of links $\bar{L}_k$ in graph G_k at any discrete time $k \in \{1, 2, \dots, K\}$ is, with $p = \sum_{l=1}^N \tilde{s}_l^2 \in [\frac{1}{N}, 1]$,

$$\bar{L}_k = \frac{M(M-1)}{2} p \in \left[\frac{M(M-1)}{2N}, \frac{M(M-1)}{2} \right].$$

- 2) The joint probability of a link (i, j) at discrete time $k+\kappa$ and k in the steady-state is

$$\Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1] = \tilde{s} \text{diag}(P^\kappa (P^\kappa)^T) \tilde{s}^T \in \left[\frac{1}{N^2}, 1 \right].$$

Proof. Equation (8) enables the computation of the expected number of links in a temporal graph at discrete time k . Since the probability of a link between any pairs of walkers is p , the expected number of links $\bar{L}_k$ in graph G_k with M walkers at any discrete time $k \in \{1, 2, \dots, K\}$ is, with $p \in [\frac{1}{N}, 1]$,

$$\bar{L}_k = \frac{M(M-1)}{2} p \in \left[\frac{M(M-1)}{2N}, \frac{M(M-1)}{2} \right].$$

Theorem 1 states that the joint probability of a link (i, j) at time $k+\kappa$ and k in the steady state is

$$\Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1] = \sum_{c_0=1}^N (\tilde{s})_{c_0}^2 \sum_{c_1=1}^N [(P^\kappa)_{c_0 c_1}]^2.$$

The matrix P^κ is stochastic with $0 \leq (P^\kappa)_{c_0 c_1} \leq 1$ and $\sum_{c_1=1}^N (P^\kappa)_{c_0 c_1} = 1$ for each row c_0 . Similarly to relation (8), we apply the Cauchy-Schwarz inequality and obtain $\sum_{c_1=1}^N ((P^\kappa)_{c_0 c_1})^2 \in [\frac{1}{N}, 1]$. The lower and the upper bounds of $\Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1]$ are given by

$$\Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1] \geq \frac{1}{N} \sum_{c_0=1}^N (\tilde{s})_{c_0}^2 \geq \frac{1}{N^2}$$

and

$$\Pr[a_{ij}[k+\kappa]=1, a_{ij}[k]=1] \leq \sum_{c_0=1}^N (\tilde{s})_{c_0}^2 \leq 1,$$

because $\sum_{c_0=1}^N (\tilde{s})_{c_0}^2 \in [\frac{1}{N}, 1]$. Thus, the joint probability $\Pr[a_{ij}[k+\kappa]=1 | a_{ij}[k]=1] \in [\frac{1}{N^2}, 1]$ for any integer $\kappa > 0$. $\square$

C.2 MSE_{link} of RWIG in the steady state where M walkers have the same steady-state vector $\tilde{s}$

Consider the RWIG model where all M walkers have the same Markov transition matrix P with an arbitrary number N of states, which admits an $1 \times N$ steady-state vector $\tilde{s} = \tilde{s}P$, and $s_i[0] = \tilde{s}$ for any $i \in \{1, \dots, M\}$. Substitution of (8) into (6) gives

$$\begin{aligned} MSE_{link}(A, \hat{A}) &= \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M (a_{ij}[k] - \tilde{s} \tilde{s}^T)^2 \\ &= \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M (a_{ij}^2[k] - 2a_{ij}[k] \tilde{s} \tilde{s}^T + (\tilde{s} \tilde{s}^T)^2) \\ &= (\tilde{s} \tilde{s}^T)^2 - 2 \tilde{s} \tilde{s}^T \frac{\sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M a_{ij}[k]}{K \binom{M}{2}} + \\ &+ \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M a_{ij}^2[k] = \left(\tilde{s} \tilde{s}^T - \frac{\sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j>i}^M a_{ij}[k]}{K \binom{M}{2}} \right)^2 \\ &- \frac{\left(\sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j>i}^M a_{ij}[k] \right)^2}{K^2 \binom{M}{2}^2} + \frac{\sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j>i}^M a_{ij}^2[k]}{K \binom{M}{2}} \\ &= \left(\tilde{s} \tilde{s}^T - \bar{a} \right)^2 + \frac{\sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j>i}^M a_{ij}^2[k]}{K \binom{M}{2}} \\ &- \frac{\left(\sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j>i}^M a_{ij}[k] \right)^2}{K^2 \binom{M}{2}^2} = \left(\tilde{s} \tilde{s}^T - \bar{a} \right)^2 + b. \end{aligned}$$

where $\bar{a}$ denotes the average number of links in a temporal graph sequence $G_0, \dots, G_{K-1}$.

C.3 Proof of Theorem 2

Theorem 2. Consider RWIG where all M walkers have the same Markov matrix P with N states, which admits an $1 \times N$ steady-state vector $\tilde{s} = \tilde{s}P$, and $s_i[0] = \tilde{s}$ for any $i \in \{1, \dots, M\}$. Any sequence $A = (A[0], \dots, A[K-1])$ of $M \times M$ adjacency matrices with an average graph density $\bar{a}$ can be reproduced by RWIG with

- 1) $MSE_{link}(A, \hat{A}) = \frac{\sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M (a_{ij}[k] - \frac{1}{N})^2}{K \binom{M}{2}} > 0$ if $\bar{a} < \frac{1}{N}$. The steady-state vector is $\tilde{s} = \frac{1}{N}$.
- 2) $MSE_{link}(A, \hat{A}) = \frac{\text{Var}[A]}{K \binom{M}{2}} \geq 0$ if $\bar{a} \geq \frac{1}{N}$. The steady-state vector $\tilde{s}$ satisfies $\tilde{s} \tilde{s}^T = \bar{a}$ and has finite solutions if and only if $\bar{a} \in \{\frac{1}{N}, 1\}$.

The minimal number of Markov states N to achieve the lowest MSE_{link} is $N = \lceil 1/\bar{a} \rceil$.

Proof. The global minimum of MSE_{link} for an arbitrary vector $\tilde{s}$ occurs at

$$\tilde{s} \tilde{s}^T = \sum_{l=1}^N \tilde{s}_l^2 = \bar{a}. \quad (29)$$

Geometrically, $\sum_{l=1}^N \tilde{s}_l = 1$ represents a hyperplane in the N -dimensional Euclidean space, while $\sum_{l=1}^N \tilde{s}_l^2 = c^2$ is a sphere with radius $c \in \mathbb{R}_{\geq 0}$. Hence, the system of equations

$$\begin{cases} \sum_{l=1}^N \tilde{s}_l^2 = c^2 \\ \sum_{l=1}^N \tilde{s}_l = 1 \\ \tilde{s}_l \geq 0, \ l \in \{1, \dots, N\} \end{cases}$$

has

- a unique solution for $c = \frac{1}{\sqrt{N}}$ when a hyperplane touches a given N -sphere at $\tilde{s} = (\frac{1}{N}, \dots, \frac{1}{N})$;
- N solutions for $c = 1$ when a hyperplane meets a given N -sphere in N axes points;
- infinitely many solutions for $c \in (\frac{1}{\sqrt{N}}, 1)$ when a hyperplane crosses a given N -sphere;
- no solution for $c < \frac{1}{\sqrt{N}}$ or $c > 1$ when a hyperplane does not meet a given N -sphere in the non-negative quadrant.

Therefore, equation (9) has a solution if and only if $\bar{a} \in [\frac{1}{N}, 1]$ because $\sum_{l=1}^N \tilde{s}_l^2 \in [\frac{1}{N}, 1]$. However, since $\lim_{N \rightarrow \infty} \sum_{l=1}^N \tilde{s}_l^2 = \lim_{N \rightarrow \infty} \frac{1}{N} = 0$ for $\tilde{s} = \frac{u}{N}$, equation (9) always has a solution for any $\bar{a} \in [0, 1]$. $\square$

C.4 MSE_{link} of RWIG in the steady state where M walkers have steady-state vectors $\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_M$

Consider the RWIG model where M walkers have $N \times N$ Markov transition matrices $P_1, P_2, \dots, P_M$ with an arbitrary number N of states, which admit $1 \times N$ steady-state vectors $\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_M$ and $s_i[0] = \tilde{s}_i$ for any $i \in \{1, \dots, M\}$. Substitution of (16) into (6) gives

$$\begin{aligned}
 MSE_{link}(A, \hat{A}) &= \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M \left(a_{ij}[k] - \tilde{s}_i \tilde{s}_j^T \right)^2 \\
 &= \frac{1}{K \binom{M}{2}} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M \left(a_{ij}^2[k] - 2a_{ij}[k] \tilde{s}_i \tilde{s}_j^T + (\tilde{s}_i \tilde{s}_j^T)^2 \right) \\
 &= \frac{1}{K \binom{M}{2}} \sum_{i=1}^M \sum_{j=i+1}^M \left(\sum_{k=0}^{K-1} a_{ij}^2[k] - 2 \sum_{k=0}^{K-1} a_{ij}[k] \tilde{s}_i \tilde{s}_j^T + K (\tilde{s}_i \tilde{s}_j^T)^2 \right) \\
 &= \frac{1}{\binom{M}{2}} \sum_{i=1}^M \sum_{j=i+1}^M \left(\frac{\sum_{k=0}^{K-1} a_{ij}^2[k]}{K} - 2 \frac{\sum_{k=0}^{K-1} a_{ij}[k]}{K} \tilde{s}_i \tilde{s}_j^T + (\tilde{s}_i \tilde{s}_j^T)^2 \right) \\
 &= \frac{1}{\binom{M}{2}} \sum_{i=1}^M \sum_{j=i+1}^M \left(\tilde{s}_i \tilde{s}_j^T - \frac{\sum_{k=0}^{K-1} a_{ij}[k]}{K} \right)^2 - \left(\frac{\sum_{k=0}^{K-1} a_{ij}^2[k]}{K} \right)^2 \\
 &\quad + \frac{\sum_{k=0}^{K-1} a_{ij}^2[k]}{K} = \frac{1}{\binom{M}{2}} \sum_{i=1}^M \sum_{j=i+1}^M \left(\tilde{s}_i \tilde{s}_j^T - \bar{a}_{ij} \right)^2 \\
 &\quad - \frac{1}{\binom{M}{2}} \sum_{i=1}^M \sum_{j=i+1}^M \bar{a}_{ij}^2 + \frac{\sum_{i=1}^M \sum_{j=i+1}^M \sum_{k=0}^{K-1} a_{ij}^2[k]}{K \binom{M}{2}} = \\
 &= \frac{1}{\binom{M}{2}} \sum_{i=1}^M \sum_{j=i+1}^M \left(\tilde{s}_i \tilde{s}_j^T + \bar{a}_{ij} \right)^2 + b,
 \end{aligned}$$

where $\bar{a}_{ij}$ is the average number of links between walkers i and j over K time slots.



Fig. 6. PATH rail system.

C.5 MSE_{clique} of RWIG in the steady state where M walkers have the same steady-state vector $\tilde{s}$

Consider the RWIG model where all M walkers have the same Markov transition matrix P with an arbitrary number N of states, which admits an $1 \times N$ steady-state vector $\tilde{s} = \tilde{s}P$, and $s_i[0] = \tilde{s}$ for any $i \in \{1, \dots, M\}$. Substitution of (10) into (7) gives

$$MSE_{clique}(A, \hat{A}) = \sum_{r=2}^M \frac{\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} (\mathbb{I}_C[k] - \sigma_r)^2}{K \binom{M}{r}},$$

where $r = |C|$. We rewrite the numerator of $MSE_{clique}(A, \hat{A})$

$$\begin{aligned}
 \sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} (\mathbb{I}_C[k] - \sigma_r)^2 &= \sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} (\mathbb{I}_C[k] - 2\sigma_r \mathbb{I}_C[k] + \sigma_r^2) \\
 &= K \binom{M}{r} \sigma_r^2 - 2\sigma_r^2 \left(\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k] \right) + \sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k] \\
 &= K \binom{M}{r} \left(\sigma_r - \frac{\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k]}{K \binom{M}{r}} \right)^2 + \sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k] \\
 &\quad - \frac{\left(\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k] \right)^2}{K \binom{M}{r}} \\
 &= K \binom{M}{r} \left(\sigma_r - \frac{\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k]}{K \binom{M}{r}} \right)^2 + b_r,
 \end{aligned}$$

where $b_r \geq 0$ is a constant. Hence, $MSE_{clique}(A, \hat{A})$ becomes

$$MSE_{clique}(A, \hat{A}) = \sum_{r=2}^M (\sigma_r - q_r)^2 + \sum_{r=2}^M \frac{b_r}{K \binom{M}{r}},$$

where $q_r = \frac{\sum_{k=0}^{K-1} \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k]}{K \binom{M}{r}}$ denotes the probability of cliques of size r in $G_0, \dots, G_{K-1}$.

APPENDIX D EXPERIMENTS

D.1 Example on empirical network

Suppose that the underlying graph G represents the PATH rail system⁸ (see Figure 6) with $N = 13$ nodes and 14 links ($L = 28$ directed links). We consider $M = L$ independent

8. <https://www.panynj.gov/path/en/schedules-maps.html>

random walkers that traverse the underlying graph G with respect to the same $N \times N$ probability transition matrix P .

Assume that the generated K -length contact sequence $G_0, \dots, G_{K-1}$ is generated in the steady state and the steady-state clique probabilities are

$$\begin{aligned} \begin{matrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \\ \sigma_8 \\ \sigma_9 \\ \sigma_{10} \\ \sigma_{11} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \\ \sigma_{15} \\ \sigma_{16} \\ \sigma_{17} \\ \sigma_{18} \\ \sigma_{19} \\ \sigma_{20} \\ \sigma_{21} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{24} \\ \sigma_{25} \\ \sigma_{26} \\ \sigma_{27} \\ \sigma_{28} \end{matrix} &= \begin{bmatrix} 1 \\ 1.19 \times 10^{-1} \\ 1.60 \times 10^{-2} \\ 2.33 \times 10^{-3} \\ 3.58 \times 10^{-4} \\ 5.70 \times 10^{-5} \\ 9.31 \times 10^{-6} \\ 1.55 \times 10^{-6} \\ 2.60 \times 10^{-7} \\ 4.40 \times 10^{-8} \\ 7.49 \times 10^{-9} \\ 1.28 \times 10^{-9} \\ 2.19 \times 10^{-10} \\ 3.76 \times 10^{-11} \\ 6.46 \times 10^{-12} \\ 1.11 \times 10^{-12} \\ 1.91 \times 10^{-13} \\ 3.30 \times 10^{-14} \\ 5.68 \times 10^{-15} \\ 9.81 \times 10^{-16} \\ 1.69 \times 10^{-16} \\ 2.92 \times 10^{-17} \\ 5.05 \times 10^{-18} \\ 8.74 \times 10^{-19} \\ 1.51 \times 10^{-19} \\ 2.62 \times 10^{-20} \\ 4.53 \times 10^{-21} \\ 7.84 \times 10^{-22} \end{bmatrix}, \quad \begin{matrix} \sigma_{1,1} \\ \sigma_{2,1} \\ \sigma_{3,1} \\ \sigma_{4,1} \\ \sigma_{5,1} \\ \sigma_{6,1} \\ \sigma_{7,1} \\ \sigma_{8,1} \\ \sigma_{9,1} \\ \sigma_{10,1} \\ \sigma_{11,1} \\ \sigma_{12,1} \\ \sigma_{13,1} \\ \sigma_{14,1} \\ \sigma_{15,1} \\ \sigma_{16,1} \\ \sigma_{17,1} \\ \sigma_{18,1} \\ \sigma_{19,1} \\ \sigma_{20,1} \\ \sigma_{21,1} \\ \sigma_{22,1} \\ \sigma_{23,1} \\ \sigma_{24,1} \\ \sigma_{25,1} \\ \sigma_{26,1} \\ \sigma_{27,1} \\ \sigma_{28,1} \end{matrix} = \begin{bmatrix} 1 \\ 6.27 \times 10^{-2} \\ 4.79 \times 10^{-3} \\ 4.19 \times 10^{-4} \\ 4.03 \times 10^{-5} \\ 4.13 \times 10^{-6} \\ 4.38 \times 10^{-7} \\ 4.74 \times 10^{-8} \\ 5.18 \times 10^{-9} \\ 5.71 \times 10^{-10} \\ 6.30 \times 10^{-11} \\ 6.98 \times 10^{-12} \\ 7.73 \times 10^{-13} \\ 8.57 \times 10^{-14} \\ 9.50 \times 10^{-15} \\ 1.05 \times 10^{-15} \\ 1.17 \times 10^{-16} \\ 1.30 \times 10^{-17} \\ 1.44 \times 10^{-18} \\ 1.60 \times 10^{-19} \\ 1.77 \times 10^{-20} \\ 1.96 \times 10^{-21} \\ 2.18 \times 10^{-22} \\ 2.42 \times 10^{-23} \\ 2.68 \times 10^{-24} \\ 2.98 \times 10^{-25} \\ 3.30 \times 10^{-26} \\ 3.66 \times 10^{-27} \end{bmatrix} \end{aligned}$$

By using Lemma 3, the number of nodes (states) in the underlying Markov graph is $N=13$. From (15), the steady-state vector is given by $\tilde{s} = [0.175, 0.1662, 0.1175, 0.1109, 0.1038, 0.1029, 0.0739, 0.0711, 0.0517, 0.0197, 0.0032, 0.0031, 0.001]$.

By using Theorem 4, we identify $L=28$ from the joint probabilities $\sigma_{1,1}, \dots, \sigma_{M,1}$. We construct the polynomial $p(L, z)$ of order L , whose positive roots are $z = [0.1109, 0.1109, 0.0662, 0.0662, 0.0623, 0.0586, 0.0580, 0.0515, 0.0514, 0.0514, 0.0498, 0.0490, 0.0382, 0.0377, 0.0377, 0.0197, 0.0197, 0.0158, 0.0153, 0.0125, 0.0071, 0.0022, 0.0022, 0.0017, 0.0010, 0.0010, 0.0009, 0.0009]$. The solution of the placement problem in (22) provides only two possible arrangements or roots $z_1, \dots, z_{28}$ in the W matrix that correspond to matrices P_1 and P_2 which are defined by

$$P_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.31 & 0.69 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.74 & 0.24 \\ 0 & 0 & 0 & 0.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0.72 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.79 & 0 & 0 & 0 & 0.21 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.15 & 0 & 0.37 & 0 & 0 & 0.48 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.36 & 0 & 0 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.44 & 0 & 0 & 0.56 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0.35 & 0.04 & 0 & 0 & 0 & 0 & 0.31 & 0 \\ 0 & 0 & 0 & 0 & 0.01 & 0 & 0 & 0 & 0 & 0.63 & 0 & 0.36 & 0 & 0 \end{bmatrix}$$

and

$$P_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.31 & 0.69 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.95 & 0.03 & 0 \\ 0 & 0 & 0 & 0.28 & 0 & 0 & 0 & 0 & 0 & 0 & 0.72 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.21 & 0 & 0 & 0 & 0.79 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.56 & 0 & 0.37 & 0 & 0 & 0.07 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.36 & 0 & 0 & 0.64 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.44 & 0 & 0 & 0.56 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.23 & 0 & 0.1 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0.37 \\ 0 & 0 & 0 & 0 & 0.07 & 0 & 0 & 0 & 0 & 0 & 0.63 & 0 & 0.29 & 0 \end{bmatrix}$$

Figure 6 illustrates the mapping between the nodes in the real underlying graph G and their labels as represented in matrices P_1 and P_2 . The difference between two matrices P_1 and P_2 is highlighted in blue. Matrix P_1 corresponds to the original Markov chain, which was applied to define the policy of each walker and generate the initial K -length contact sequence. The second Markov chain is the reversed Markov chain. We would like to emphasize that both Markov chains have the same underlying graph G , which corresponds to the Path rail system. Therefore, experiment demonstrates that it is possible to infer the initial Markov process from a given time series of contact graphs.

D.2 Inferring the steady-state vector from a single realization

In our experiments, we assume that the observed contact sequence $G_0, \dots, G_{K-1}$ is generated in the steady-state. Hence the empirical clique probabilities equal the steady-state probabilities. In this Section, we discuss how the length of the sequence K and the number of walkers M affects the accuracy of empirical clique probabilities and, consequently, the accuracy of the proposed methodology.

Example 4. Consider the Markov graph with $N=4$ states and $M=5$ walkers that traverse the Markov graph in discrete time according to the 4×4 transition probability matrix

$$P = \begin{bmatrix} 0.48 & 0.04 & 0 & 0.48 \\ 0.21 & 0.25 & 0.26 & 0.28 \\ 0.14 & 0.25 & 0.34 & 0.27 \\ 0.42 & 0.38 & 0.08 & 0.12 \end{bmatrix}.$$

Suppose that the initial states of the walkers $w_1, \dots, w_5$ are

$$X_1[0] = X_2[0] = X_5[0] = 0, \quad X_3[0] = 3, \quad X_4[0] = 2.$$

We perform a single realization of RWIG. At each step k walker w_i selects the next state $X_i[k+1]$ based on the current state $X_i[k]$ and the probability matrix P . For each discrete time k , we estimate the probability q_r of the clique of size r ($r \leq M$ based on the observed sequence of adjacency matrices $A[0], \dots, A[k]$) as

$$q_r = \frac{\sum_{k=0}^K \sum_{C \in \mathcal{M}_r} \mathbb{I}_C[k]}{K \binom{M}{r}}.$$

Given the steady-state clique probabilities $q_1, \dots, q_M$, we evaluate the components of the steady-state vector $\tilde{s}$ using (15). Figure 7 demonstrates the possibility to recover the steady-state vector $\tilde{s}$ from a single realization. Since the Markov chain with matrix P is irreducible, aperiodic and positive recurrent, the state vectors of M walkers will converge to the unique steady-state vector $\tilde{s}$. As the

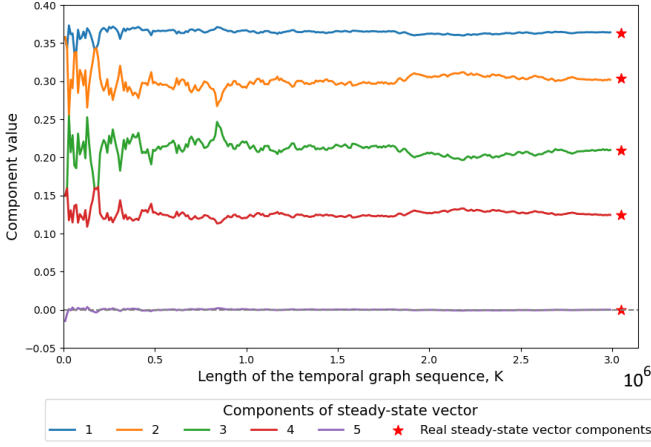


Fig. 7. The steady-state vector from one realization of RWIG.

length K of the graph sequence $G_0, \dots, G_{K-1}$ increases, the estimated probabilities $q_2, \dots, q_M$ from K graphs tend closer to the real steady-state probabilities $\sigma_2, \dots, \sigma_M$, which are defined in (10). In our example, the estimated steady-state vector converges to the real steady-state vector $\tilde{s}$ for $K > 800,000$, because, for smaller K , the number of observations is insufficient to capture the probability of a clique of size 4, which is rarely observed in the contact sequence ($\sigma_4 \approx 0.028$). Figure 7 illustrates that the Markov chain has $N=4$ states because $a_5 \approx 0$ for large K and, consequently, $p_M(z)=0$ gives $z_5 = 0$. The estimated steady-state vector is $\tilde{s} = [0.363, 0.303, 0.209, 0.125]$.

We would like to point out that the length of the graph sequence K does not necessarily need to satisfy $K > 800,000$ to recover the initial steady-state vector $\tilde{s}$. By the law of large numbers, as the number of observations increases, the estimated probabilities of cliques become more accurate, steadily converging to the real steady-state values σ_r . Figure 8 illustrates how the L1-norm between the estimated vector $\hat{s}$ and the steady-state vector $\tilde{s}$ varies with the sequence length K and the number of random walkers M . We observe that increasing the number of random walkers M can significantly reduce the required length K of the contact sequence. For instance, if $M = 20$, then the estimated steady-state vector $\hat{s}$ converges to the real steady-state vector $\tilde{s}$ after $K = 700$. In many practical applications, such as transportation networks, the number of walkers is often much greater than the number of states ($M \gg N$). Therefore, recovering the initial process is achievable.

APPENDIX E

RWIG WITH DIFFERENT STOCHASTIC MATRICES IN THE STEADY STATE

E.1 Proof of Lemma 4

Lemma 4. Consider RWIG where M walkers have $N \times N$ Markov transition matrices $P_1, \dots, P_M$ with an arbitrary number N of states, which admit $1 \times N$ steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$ and $s_i[0] = \tilde{s}_i$ for any $i \in \{1, \dots, M\}$. Then any sequence of graphs $G_0, \dots, G_{K-1}$ that do not change over time, where G_0 consists of the union of m disconnected cliques, can be accurately modelled by RWIG in the steady state where $N \geq m$.

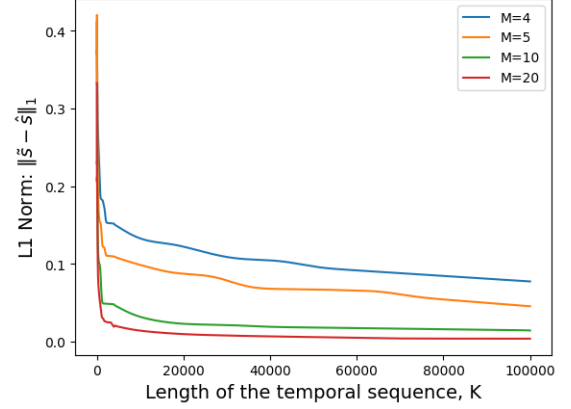


Fig. 8. The accuracy of the steady-state vector (L1-norm) from one realization of RWIG.

Proof. If the temporal graph contains m disconnected cliques that do not change over time, then $\bar{a}_{ij} = 1$ if walkers w_i and w_j belong to the same clique, otherwise $\bar{a}_{ij} = 0$. Then one possible solution of (18) is a collection of $1 \times m$ steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$ with $\tilde{s}_i = \delta_{i c_i}$ where c_i is the membership of walker w_i in one of the cliques. The Markov graph contains $N=m$ states and $MSE_{link}(A, \hat{A})=0$. $\square$

E.2 Proof of Theorem 3

Theorem 3. Let $\mathcal{G}$ be an $M \times M$ symmetric matrix with $g_{ij} \in [0, 1]$ for any $i \neq j$ and $\mathcal{G} = U \Lambda U^T$ be the eigenvalue decomposition of $\mathcal{G}$ where $U = [u_1 \ u_2 \ \dots \ u_M]$ is an $M \times M$ orthogonal matrix formed by the scaled, real eigenvalues u_k belonging to eigenvalue $\lambda_k(\mathcal{G})$, $\Lambda = \text{diag}(\lambda_k(\mathcal{G}))$ is an $M \times M$ diagonal matrix of eigenvalues and $\lambda_1(\mathcal{G}) \geq \dots \geq \lambda_M(\mathcal{G})$. Then $\mathcal{G}$ is the Gram matrix of $1 \times N$ steady-state vectors $\tilde{s}_1, \dots, \tilde{s}_M$ if and only if

- 1) $\mathcal{G}$ is positive semidefinite;
- 2) $g_{ii} \in [\frac{1}{N}, 1]$;
- 3) $N > \text{rank}(\tilde{X})$, where $\tilde{X}$ is an $(M-1) \times M$ matrix with $\tilde{X} = [x_2 - x_1 \ \dots \ x_M - x_1]^T$ and $X = U \Lambda^{1/2} = [x_1 \ \dots \ x_M]^T$;
- 4) there exists an $N \times 1$ non-zero vector n such that

$$\begin{cases} \sum_{j=1}^{\text{rank}(\mathcal{G})} n_j (\tilde{X})_{ij} = 0, \forall i \in \{1, \dots, M-1\}, \\ \frac{|\sum_{j=1}^{\text{rank}(\mathcal{G})} n_j (X)_{1j}|}{\|n\|} = \frac{1}{\sqrt{N}}. \end{cases}$$

- 5) there exists an $N \times N$ orthogonal matrix R such that $X R = [\tilde{s}_1 \ \tilde{s}_2 \ \dots \ \tilde{s}_M]$ and $\frac{1}{\sqrt{N}\|n\|} R n = \frac{u}{N}$.

Proof.

- 1) Positive semidefiniteness is a property of the Gram matrix $\mathcal{G}$ that ensures that all inner products are consistent with the notion of angle and distance in Euclidean space. The Gram matrix also imposes an additional constraint on its diagonal elements $g_{ii} = \tilde{s}_i \tilde{s}_i^T$. For instance, the non-negativity condition for the second-order principal minors

$$\begin{vmatrix} g_{ii} & \bar{a}_{ij} \\ \bar{a}_{ij} & g_{jj} \end{vmatrix} \geq 0$$

for $j \neq i$ implies that $g_{ii} \geq \frac{\bar{a}_{ij}^2}{g_{jj}} \geq \bar{a}_{ij}^2$ because $g_{jj} \in [\frac{1}{N}, 1]$. Hence, $g_{ii} \geq \max_j \bar{a}_{ij}^2$.

- 2) The diagonal element g_{ii} of the Gram matrix $\mathcal{G}$ is the squared length of the steady-state vector $\tilde{s}_i$ on the $(N-1)$ -dimensional simplex. Hence, g_{ii} varies from $\frac{1}{N}$ to 1.
- 3) The rank of $\mathcal{G}$ indicates the number of linearly independent steady-state vectors $\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_M$. By definition, $1 \times N$ steady-state vectors belong to an $(N-1)$ -dimensional simplex. In addition, vectors $\tilde{s}_2 - \tilde{s}_1, \dots, \tilde{s}_M - \tilde{s}_1$ lie on the $(N-1)$ -dimensional hyperplane with normal vector u and, consequently, $\text{rank}([\tilde{s}_2 - \tilde{s}_1 \dots \tilde{s}_M - \tilde{s}_1]) < N$.

The eigenvalue decomposition of the $M \times M$ Gram matrix $\mathcal{G} = U\Lambda U^T = (U\Lambda^{1/2})^T (U\Lambda^{1/2})^T = XX^T$ defines the $M \times M$ matrix X , where the rows of $X = [x_1 \ x_2 \ \dots \ x_M]^T$ encode the transformed positions of the steady-state vectors in $\text{rank}(\mathcal{G})$ -dimensional subspace⁹. The steady-state vectors do not belong to any $(N-1)$ -dimensional simplex where $N < \text{rank}(\mathcal{G})$. Then $N \geq \text{rank}(\mathcal{G}) = \text{rank}(X)$.

Since an orthogonal transformation preserves the original geometric structure of the steady-state vectors $\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_M$ (their lengths and mutual angles), the vectors of the $(M-1) \times M$ matrix $\tilde{X} = [x_2 - x_1 \ \dots \ x_M - x_1]^T$ should also lie on the $(N-1)$ -dimensional hyperplane, which is possible if and only if $\text{rank}(\tilde{X}) < N$.

- 4) The vectors $\tilde{s}_2 - \tilde{s}_1, \dots, \tilde{s}_M - \tilde{s}_1$ belong to the $(N-1)$ -dimensional hyperplane with normal vector u at a distance of $\frac{1}{\sqrt{N}}$ from the origin. Since an orthogonal transformation preserves the original geometric structure of the steady-state vectors, the vectors $x_2 - x_1, \dots, x_M - x_1$ must belong to the hyperplane with a non-zero normal vector n which is at distance $\frac{1}{\sqrt{N}}$ from the origin. We define the system

$$\begin{cases} \sum_{j=1}^{\text{rank}(\mathcal{G})} n_j (\tilde{X})_{ij} = 0, \forall i \in \{1, \dots, M-1\}, \\ \frac{|\sum_{j=1}^{\text{rank}(\mathcal{G})} n_j (X)_{1j}|}{\|n\|} = \frac{1}{\sqrt{N}}. \end{cases} \quad (30)$$

The first equation of (30) identifies the normal vector n of the hyperplane that contains $x_2 - x_1, \dots, x_M - x_1$. The second equation of (30) verifies the distance condition where the left part of the equation evaluates the distance from the origin to the hyperplane where $\sum_{j=1}^{\text{rank}(\mathcal{G})} n_j (X)_{1j}$ defines the constant term of the hyperplane equation).

- 5) Condition 4 ensures that there exists an orthogonal transformation R that rotates the vectors in the rows of X onto an $(N-1)$ -dimensional hyperplane with normal vector u , which is at a distance of $\frac{1}{\sqrt{N}}$ from the origin. The vector $\frac{n}{\sqrt{N}\|n\|}$, which is the perpendicular to the hyperplane that contains the vectors in the rows of X , will be rotated to the center of the $(N-1)$ -dimensional simplex after applying R to the perpendicular. Hence, if all columns of XR lie within the convex hull formed by the vertices of the standard $(N-1)$ -dimensional simplex, then $\mathcal{G}$ is the Gram matrix of $1 \times N$ steady-state vectors $S = [\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_M]^T$ where $\tilde{S} = XR$.

9. The vectors in the rows of X , which form $\text{rank}(\mathcal{G})$ -dimensional subspace, can be extended to N -dimensional space by appending $N - \text{rank}(\mathcal{G})$ zeros to each of these vectors.

Identifying the appropriate orthogonal matrix R for arbitrary N and M is challenging and requires additional research¹⁰. $\square$

E.3 RWIG with different stochastic matrices $P_1, \dots, P_M$ in the steady state: the solution for $M = 2$ and $M = 3$ ($N = 2$)

We provide the solution of system (18) for $M = 2$ and $M = 3$. Lemma 8 demonstrates that RWIG in the steady state, where $M = 2$ walkers may have different Markov transition probability matrices P_1 and P_2 , is able to reproduce accurately only a sequence of complete graphs (2 walkers have a contact at any discrete time k). Any other sequence of contacts between two walkers can be reproduced with lowest possible $MSE(A, \hat{A}) = \frac{\text{Var}(A)}{K}$ on the Markov graph with $N = 2$ states where one of the walkers always remains in the same state, while another walker visits this state with probability $\bar{a}_{12}$. Lemma 9 provides the conditions on the initial sequence of contacts such that system (18) has a solution for $N = 2$.

Lemma 8. Consider the RWIG model where $M=2$ walkers have traverse the Markov graph in discrete timesteps according to the $N \times N$ Markov transition probability matrices P_1 and P_2 , which admit $1 \times N$ steady-state vectors $\tilde{s}_1$ and $\tilde{s}_2$. Then any sequence $A = (A_0, \dots, A_{K-1})$ of 2×2 adjacency matrices, which describe contacts between $M=2$ walkers over K time slots, can be approximated by RWIG with the lowest mean squared error

$$MSE(A, \hat{A}) = \frac{1}{K} \sum_{k=0}^{K-1} (a_{12}[k] - \bar{a}_{12})^2 = \frac{\text{Var}[A]}{K},$$

where $\bar{a}_{12} = \frac{1}{K} \sum_{k=0}^{K-1} a_{12}[k]$.

For any $\bar{a}_{12} \neq 1$ the minimal number of states N is 2 where $\tilde{s}_1 = (1, 0)$, $\tilde{s}_2 = (\bar{a}_{12}, 1 - \bar{a}_{12})$. If $\bar{a}_{12} = 1$, then $MSE(A, \hat{A}) = 0$ and the minimal number of states N is 1 where $\tilde{s}_1 = \tilde{s}_2 = 1$.

Proof. The system (18) for $M = 2$ walkers and $N = 2$ states can be rewritten as

$$\begin{cases} \tilde{s}_1 \tilde{s}_2^T = (\tilde{s}_1)_1 (\tilde{s}_2)_1 + (\tilde{s}_1)_2 (\tilde{s}_2)_2 = \frac{1}{K} \sum_{k=1}^K a_{12}[k] = \bar{a}_{12}, \\ (\tilde{s}_1)_1 + (\tilde{s}_1)_2 = 1, \\ (\tilde{s}_2)_1 + (\tilde{s}_2)_2 = 1, \\ \tilde{s}_1, \tilde{s}_2, \tilde{s}_3 \geq 0. \end{cases} \quad (31)$$

One possible solution of (31) is $\tilde{s}_1 = (1, 0)$, $\tilde{s}_2 = (\bar{a}_{12}, 1 - \bar{a}_{12})$ where $N = 2$. For $N = 1$ the system (18) has a solution if and only if $\tilde{s}_1 = \tilde{s}_2 = 1$ and $\bar{a}_{12} = 1$. $\square$

Lemma 9. Consider the RWIG model where $M = 3$ walkers traverse the Markov graph in discrete timesteps according to the 2×2 Markov transition probability matrices P_1 , P_2 and P_3 , which admit 1×2 steady-state vectors $\tilde{s}_1$, $\tilde{s}_2$ and $\tilde{s}_3$. Then the sequence $A = (A_0, \dots, A_{K-1})$ of 3×3 adjacency matrices, which describe contacts between $M = 3$ walkers over K time slots, can be approximated by RWIG with the lowest mean squared error

10. Condition 5 can be reformulated as follows: given the $(N-1)$ -dimensional hyperplane with normal vector n and point $P = \frac{n}{\sqrt{N}\|n\|}$, it is possible to construct an $(N-1)$ -dimensional simplex around P on the hyperplane that contains the convex hull, which is formed by the rows of X ?

$$MSE(A, \hat{A}) = \frac{1}{3K} \sum_{k=0}^{K-1} \sum_{i=1}^3 \sum_{j=i+1}^3 (a_{ij}[k] - \bar{a}_{ij})^2,$$

where $\bar{a}_{ij} = \frac{1}{K} \sum_{k=0}^{K-1} a_{ij}[k]$ if and only if one of the conditions (a)-(c) is satisfied:

- a) $\bar{a}_{12} = \bar{a}_{13} = \bar{a}_{23} = 0.5$. Then at least two walkers must have the same steady-state vector $(0.5, 0.5)$.
- b) $\exists i \neq j \neq k \in \{1, 2, 3\}$: $\bar{a}_{ij} = \bar{a}_{ik} = 0.5$ and $\bar{a}_{jk} \neq 0.5$. The steady-state vectors are $\tilde{s}_i = (0.5, 0.5)$, $\tilde{s}_j = (1, 0)$ and $\tilde{s}_k = (\bar{a}_{jk}, 1 - \bar{a}_{jk})$.
- c) $\begin{cases} 0 \leq d \leq 1, \\ -1 \leq \frac{(\bar{a}_{12}-0.5)}{\sqrt{d}} \leq 1, \\ -1 \leq \frac{(\bar{a}_{13}-0.5)}{\sqrt{d}} \leq 1, \end{cases}$ where $d = \frac{2\bar{a}_{12} + 2\bar{a}_{13} - 4\bar{a}_{12}\bar{a}_{13} - 1}{1 - 2\bar{a}_{23}}$. The steady-state vectors of the walkers are
 - $\tilde{s}_1 = (\frac{1}{2} \pm \frac{1}{2}\sqrt{d}, \frac{1}{2} \mp \frac{1}{2}\sqrt{d})$,
 - $\tilde{s}_2 = (\frac{1}{2} \pm \frac{(\bar{a}_{12}-0.5)}{\sqrt{d}}, \frac{1}{2} \mp \frac{(\bar{a}_{12}-0.5)}{\sqrt{d}})$,
 - $\tilde{s}_3 = (\frac{1}{2} \pm \frac{(\bar{a}_{13}-0.5)}{\sqrt{d}}, \frac{1}{2} \mp \frac{(\bar{a}_{13}-0.5)}{\sqrt{d}})$.

Proof. Let $\tilde{s}_i = (x_i, 1 - x_i)$ where $x_i \in [0, 1]$ for $i \in \{1, 2, 3\}$. The system (18) for $M = 3$ walkers and $N = 2$ states is given by

$$\begin{cases} x_1 x_2 + (1 - x_1)(1 - x_2) = \bar{a}_{12}, \\ x_1 x_3 + (1 - x_1)(1 - x_3) = \bar{a}_{13}, \\ x_2 x_3 + (1 - x_2)(1 - x_3) = \bar{a}_{23}, \\ x_1, x_2, x_3 \in [0, 1]. \end{cases} \quad (32)$$

The equation $ab + (1 - a)(1 - b) = 0.5$ implies¹¹ $a = 0.5$ or $b = 0.5$. Hence, if $\bar{a}_{12} = \bar{a}_{13} = \bar{a}_{23} = 0.5$, then $x_1 = x_2 = 0.5$ or $x_1 = x_3 = 0.5$ or $x_2 = x_3 = 0.5$. If $\exists i \neq j \neq k \in \{1, 2, 3\}$: $\bar{a}_{ij} = \bar{a}_{ik} = 0.5$ and $\bar{a}_{jk} \neq 0.5$, a solution of the system (32) is $\tilde{s}_i = (0.5, 0.5)$, $\tilde{s}_j = (1, 0)$ and $\tilde{s}_k = (\bar{a}_{jk}, 1 - \bar{a}_{jk})$. We note that $\bar{a}_{ij} = 0.5$, $\bar{a}_{ik} \neq 0.5$ and $\bar{a}_{jk} \neq 0.5$ is not possible because $\bar{a}_{ij} = 0.5$ implies $x_i = 0.5$ and $\bar{a}_{ik} = 0.5$.

Assume that $\bar{a}_{ij} \neq 0.5$ for any $i \neq j \in \{1, 2, 3\}$. Then $x_1 \neq 0.5$, $x_2 \neq 0.5$ and $x_3 \neq 0.5$. Rewrite the system (32) as

$$\begin{cases} 2x_1 x_2 + 1 - x_1 - x_2 = x_2(2x_1 - 1) + 1 - x_1 = \bar{a}_{12}, \\ 2x_1 x_3 + 1 - x_1 - x_3 = x_3(2x_1 - 1) + 1 - x_1 = \bar{a}_{13}, \\ 2x_2 x_3 + 1 - x_2 - x_3 = \bar{a}_{23}, \\ x_1, x_2, x_3 \in [0, 1]. \end{cases}$$

Then

$$x_2 = \frac{\bar{a}_{12} + x_1 - 1}{2x_1 - 1} = \frac{1}{2} + \frac{\bar{a}_{12} - 0.5}{2x_1 - 1} \quad (33)$$

and

$$x_3 = \frac{\bar{a}_{13} + x_1 - 1}{2x_1 - 1} = \frac{1}{2} + \frac{\bar{a}_{13} - 0.5}{2x_1 - 1}. \quad (34)$$

Substituting (33) and (34) into the third equation of the system (32):

$$\begin{aligned} 2 \left(\frac{\bar{a}_{12} + x_1 - 1}{2x_1 - 1} \right) \left(\frac{\bar{a}_{13} + x_1 - 1}{2x_1 - 1} \right) + 1 - \left(\frac{\bar{a}_{12} + x_1 - 1}{2x_1 - 1} \right) \\ - \left(\frac{\bar{a}_{13} + x_1 - 1}{2x_1 - 1} \right) = \bar{a}_{23}, \end{aligned}$$

11. Assume that $b \neq 0.5$. Then $ab + (1 - a)(1 - b) = a(2b - 1) - b + 1 = 0.5$ and $a = \frac{b - 0.5}{2b - 1} = 0.5$

which can be simplified to

$$(2 - 4\bar{a}_{23})x_1^2 - (2 - 4\bar{a}_{23})x_1 + (2\bar{a}_{12}\bar{a}_{13} - \bar{a}_{12} - \bar{a}_{13} - \bar{a}_{23} + 1) = 0,$$

with the solution

$$x_1 = \frac{1}{2} \pm \frac{1}{2} \sqrt{\frac{2\bar{a}_{12} + 2\bar{a}_{13} - 4\bar{a}_{12}\bar{a}_{13} - 1}{1 - 2\bar{a}_{23}}}.$$

Since $2x_1 - 1 = \pm \sqrt{\frac{2\bar{a}_{12} + 2\bar{a}_{13} - 4\bar{a}_{12}\bar{a}_{13} - 1}{1 - 2\bar{a}_{23}}}$, the solution for x_2 and x_3 is given by

$$\begin{aligned} x_2 &= \frac{1}{2} \pm (\bar{a}_{12} - 0.5) \sqrt{\frac{1 - 2\bar{a}_{23}}{2\bar{a}_{12} + 2\bar{a}_{13} - 4\bar{a}_{12}\bar{a}_{13} - 1}}, \\ x_3 &= \frac{1}{2} \pm (\bar{a}_{13} - 0.5) \sqrt{\frac{1 - 2\bar{a}_{23}}{2\bar{a}_{12} + 2\bar{a}_{13} - 4\bar{a}_{12}\bar{a}_{13} - 1}}. \end{aligned}$$

Therefore, the system (32) has a solution if $x_1, x_2, x_3 \in [0, 1]$. $\square$

APPENDIX F PROOF OF THEOREM 4

Theorem 4. Consider the $N \times N$ matrix $W = P \odot \tilde{s}^T$. Given the steady-state vector $\tilde{s}$ and the steady-state joint probability $\sigma_{r,1}$ for each integer $2 \leq r \leq N^2$, the elements of matrix W are the roots of a polynomial of order N^2

$$p_{N^2}(z) = \sum_{r=0}^{N^2} a_r z^r, \quad (35)$$

where $a_r = -\frac{1}{N^2 - r} \sum_{l=r+1}^{N^2} a_l \sigma_{l,1}$ with $a_{N^2} = 1$. The position of the r th root z_r of (21) in the matrix W is defined by the solution of the placement problem

$$\begin{cases} \sum_{r=1}^{N^2} y_{ir} z_r = \tilde{s}_i, & \text{for all } i \in \{1, \dots, N\}, \\ \sum_{r=1}^{N^2} y_{rj} z_r = \tilde{s}_j, & \text{for all } j \in \{1, \dots, N\}, \\ \sum_{j=1}^N y_{rj} = 1, & \text{for all } r \in \{1, \dots, N^2\}, \\ \sum_{i=1}^N y_{ir} = 1, & \text{for all } r \in \{1, \dots, N^2\}, \\ \sum_{r=1}^{N^2} y_{ir} = N, & \text{for all } i \in \{1, \dots, N\}, \\ \sum_{r=1}^{N^2} y_{rj} = N, & \text{for all } j \in \{1, \dots, N\} \end{cases} \quad (36)$$

where $y_{ir}, y_{rj} \in \{0, 1\}$ for all $i, j \in \{1, \dots, N\}$ are the binary variables that define if the r th root z_r , $r \in \{1, \dots, N^2\}$ has position w_{ij} in matrix W .

Proof. In general, the r th power sum of the elements of W has N^2 non-zero variables. Consider a polynomial $p_{N^2}(z)$ in N^2 variables

$$p_{N^2}(z) = \sum_{r=0}^{N^2} a_r z^r = a_N^2 \prod_{r=1}^{N^2} (z - z_r),$$

where $a_0, \dots, a_{N^2}$ are the coefficients of a polynomial $p_{N^2}(z)$ with $a_{N^2} = 1$ and $z_1, \dots, z_{N^2}$ are the roots of the polynomial and the elements of matrix W . Given the steady-state joint probability $\sigma_{r,1}$ for each integer $2 \leq r \leq N^2$, the r th power sum $\sigma_{r,1}$ is

$$\sum_{i=1}^N \sum_{j=1}^N w_{ij}^r = \sigma_{r,1}.$$

From [21], the relation between the coefficients $a_0, \dots, a_{N^2}$ and the power sums $\sigma_{1,1}, \dots, \sigma_{N,1}$ is

$$a_r = -\frac{1}{N-r} \sum_{l=r+1}^N a_l \sigma_{l,1}.$$

To recover matrix W , we need to identify the position of each root of $p(N^2, z) = 0$ in matrix W . Hence, we formulate the placement problem in (36), which aims to find the optimal arrangement of roots $\{z_r\}_{r=1}^{N^2}$ in the matrix W , subject to constraints $Wu^T = \tilde{s}^T$ and $uW = \tilde{s}$. We denote by y_{ir} and y_{rj} the binary variables that define if r th root z_r has position w_{ij} in matrix W . The total number of variables is $2N^3$ because we introduced $2N$ variables y_{ir} and y_{rj} for each root z_r , $r \in \{1, \dots, N^2\}$. The first constraint in (36) ensures $Wu^T = \tilde{s}^T$, the constraint equation ensures $uW = \tilde{s}$. Equations 3-4 in (36) guarantee that each root x_r is assigned to exactly one row and column. The remaining equations ensure that N roots are assigned to each row and column of W . $\square$

APPENDIX G

RWIG FOR PERIODIC SEQUENCES

G.1 Proof of Lemma 5

Lemma 5. *The RWIG model, where the walkers have the same $N \times N$ Markov transition matrix P , can accurately reproduce only 1-periodic graph sequences $G_0, G_1, \dots, G_{K-1}$.*

Proof. Given the sequence $A = (A_0, \dots, A_{K-1})$ of adjacency matrices over K time slots, we need to identify the initial state vectors $S_{\mathcal{M}}^*[0] = (s_1^*[0], \dots, s_M^*[0])$ of M walkers and their transition probability matrices $P_{\mathcal{M}}^* = (P_1^*, \dots, P_M^*)$ that minimize relation (6)

$$S_{\mathcal{M}}^*[0], P_{\mathcal{M}}^* = \arg \min_{s_{\mathcal{M}}^*[0], P_{\mathcal{M}}^*} \frac{1}{K} \sum_{k=0}^{K-1} \sum_{i=1}^M \sum_{j=i+1}^M \left(a_{ij}[k] - s_i[k] s_j^T[k] \right)^2. \quad (37)$$

If the graph sequence $G_0, G_1, \dots, G_{K-1}$ is 1-periodic and G_1 contains m cliques, then one possible solution of (37) is the $m \times m$ transition matrix $P = I$ and a collection of $1 \times m$ initial state vectors $s_1[0], \dots, s_M[0]$ with $s_i[0] = \delta_{i,c_i}$ where c_i is the membership of walker w_i in one of m cliques in G_1 .

Suppose that the graph sequence $G_0, G_1, \dots, G_{K-1}$ is p -periodic and $1 < p < K-1$. Then there exist two walkers w_i and w_j such that $a_{ij}[k] = 1$ at discrete time k and $a_{ij}[k+1] = 0$. Lemma 1 states that the state vectors of walkers w_i and w_j at discrete time k should be equal to each other and $s_i[k] = s_j[k] = e_l$. Since $s_i[k+1] = s_i[k]P$ for any walker w_i , the state vectors $s_i[k+1]$ and $s_j[k+1]$ of walkers i and j at discrete time $k+1$ are also equal to each other

$$s_i[k+1] = s_j[k+1] = e_l P = [P_{l1} \ P_{l2} \ \dots \ P_{lM}].$$

The probability of a contact $\hat{a}_{ij}[k+1]$ between two walkers i and j at discrete time $k+1$ given that they have a contact at time k is

$$\hat{a}_{ij}[k+1] = s_i[k+1] \cdot s_j^T[k+1] = \sum_{r=1}^N (P_{lr})^2 > 0,$$

because $0 \leq P_{lr} \leq 1$ and $\sum_{r=1}^N P_{lr} = 1$. Therefore, $\hat{a}_{ij}[k+1] \neq a_{ij}[k+1] = 0$. Lemma 5 is proven. $\square$

G.2 Proof of Lemma 6

Lemma 6. *Any p -periodic graph sequences $G_0, \dots, G_{K-1}$ can be accurately reproduced by RWIG where M walkers have different $N \times N$ Markov transition matrices $P_1, \dots, P_M$ and $N = \sum_{k=0}^{p-1} n_c[k]$, $n_c[k]$ is the number of cliques in G_k .*

Proof. Suppose that the graph sequence $G_0, G_1, \dots, G_{K-1}$ is p -periodic. We create a Markov graph with $N = \sum_{k=0}^{p-1} n_c[k]$ states $1, 2, \dots, N$ where $n_c[k]$ is the number of cliques in G_k . Each state j of the Markov graph corresponds to a clique C_j in the temporal graph at discrete time $k[j] = \min\{t : j \leq \sum_{k=0}^t n_c[k]\}$. Since the graph sequence is p -periodic and walker w_i is a member of p cliques in graphs $G_0, G_1, \dots, G_{p-1}$, we can reproduce the graph sequence by RWIG where each walker w_i sequentially traverses p states in the Markov graph. The element $(P_i)_{jl}$ of the $N \times N$ transition matrix P_i of walker w_i is 1 if w_i is a member of cliques C_j and C_l , corresponding to the adjacent periods $k[j]$ and $k[l]$, otherwise $(P_i)_{jl} = 1$

$$(P_i)_{jl} = \begin{cases} 1 & \text{if } w_i \in C_j \cap C_l, \quad (k[j] + 1) \bmod p = k[l], \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

G.3 Proof of Lemma 7

Lemma 7. *Any non-zero contact sequence between walkers w_i and w_j with period p , where p is the minimal period of the sequence, can be reproduced by RWIG where walkers w_i and w_j traverse the Markov graph with periods p_i and p_j such that*

- a) $LCD(p_1, p_2) \bmod p = 0$, where $LCD(p_1, p_2)$ is the least common denominator of p_1 and p_2 .
- b) $\min(p_1, p_2) \geq \frac{LCD(p_1, p_2)}{p} \sum_{k=0}^{p-1} a_{ij}[k]$
- c) $\nexists k_1, k_2 < LCD(p_1, p_2)$ with $a_{ij}[k_1] = a_{ij}[k_2] = 1$ such that

$$\begin{cases} k_1 \bmod p_1 = k_2 \bmod p_1, \\ k_1 \bmod p_2 = k_2 \bmod p_2. \end{cases}$$

Proof. a) Assume that walkers w_i and w_j are in states l_1 and l_2 at discrete time k . If walkers w_i and w_j traverse the Markov graph with periods p_i and p_j , then both walkers will be again in states l_1 and l_2 at discrete time $k + LCD(p_1, p_2)$. Hence, walkers w_i and w_j produce a $LCD(p_1, p_2)$ -periodic contact sequence. Suppose that $LCD(p_1, p_2) \bmod p = r$ with $0 < r < p$. Then $a_{ij}[k] = a_{ij}[k + LCD(p_1, p_2)] = a_{ij}[k + \lfloor LCD(p_1, p_2)/p \rfloor + r] = a_{ij}[k + r]$. Hence, the contact sequence is r -periodic, which is a contradiction (p is the minimal period). Thus, $LCD(p_1, p_2) \bmod p = 0$.

b) Each walker w_i traverses p_i different states in the Markov graph. Then the maximum number of shared states between walkers w_i and w_j is $\min(p_1, p_2)$. If w_i and w_j are in state l at discrete time k , then they will be again in the same state l at time $k + LCD(p_1, p_2)$. Thus, the maximum number of contacts between w_i and w_j over $LCD(p_1, p_2)$ time slots is $\min(p_1, p_2)$.

c) Suppose that there exist discrete time slots $k_1 < k_2 < LCD(p_1, p_2)$ with $a_{ij}[k_1] = a_{ij}[k_2] = 1$ such that $(k_1 \bmod p_1) = (k_2 \bmod p_1) = l_1$. Then walker w_i has two contacts with walker w_j in the same state l_1 , which is a contradiction (walkers cannot be together in the same state more than once over $LCD(p_1, p_2)$ time slots). $\square$