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# On generalized windmill graphs

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A B S T R A C T

In this paper we will consider generalizations of the class of so-called windmill graphs, which were recently introduced by Estrada [6]. A windmill graph $W(\eta, k)$ consists of $\eta$ copies of the complete graph $K_k$, with every node connected to a common node. Estrada [6] showed that the clustering coefficient and the transitivity index of windmill graphs diverge, when the graph size tends to infinity. In addition [6] studied the spectra of the adjacency and the Laplacian matrices of these graphs. In this paper we will generalize the family of windmill graphs in three ways. We will study properties for all three types of generalized windmill graphs. In particular we will focus on the behavior of the two clustering metrics. We will quantify the difference between the two metrics, under various conditions. In addition, we give the spectra of the adjacency and the Laplacian matrices of these graphs. We will also derive analytic expressions for several other graph metrics, such as average path length, heterogeneity index and a variety of robustness metrics. We also show how the generalized windmill graphs can be used to construct pairs of non-isomorphic graphs with the same number of nodes and links. Finally, we will show how

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generalized windmill graphs occur quite naturally in the study of public transportation networks.



## 1. Introduction

In this paper we will consider generalizations of the class of so-called windmill graphs, which were recently introduced by Estrada [6]. A windmill graph $W(\eta, k)$ consists of $\eta$ copies of the complete graph $K_k$, with every node connected to a common node, see Fig. 1.

Estrada [6] showed that the clustering coefficient and the transitivity index of windmill graphs diverge, when the graph size tends to infinity. In addition [6] studied the spectra of the adjacency and the Laplacian matrices of these graphs.

In this paper we will generalize the family of windmill graphs in three ways. For the first two generalizations we will replace the central node, connecting all $\eta$ copies of the complete graph $K_k$, by $l$ central nodes. For the first generalization, we assume the $l$ central nodes are all connected, i.e. they form a complete graph $K_l$. We will denote the generalized windmill graph of Type I by $W'(\eta, k, l)$. Obviously, it holds that $W'(\eta, k, 1) = W(\eta, k)$. Also, the so-called agave graphs, defined in [8], are special cases of the generalized windmill graphs of Type I. For instance, the agave graphs depicted in Fig. 2, can be represented as $W'(4, 1, 2)$ and $W'(5, 1, 2)$, respectively.

For the second generalization, we assume the $l$ central nodes have no connections among each other. We will denote this generalized windmill graph by $W''(\eta, k, l)$. Figs. 3–4 depict examples of the generalized windmill graphs of Type I and II, respectively.

For the third generalization, we assume that there are $\eta$ central nodes, such that for each of the $\eta$ copies of the complete graph $K_k$, each node connects to a different central node. The $\eta$ central nodes also form a clique. We will denote this generalized windmill graph by $W'''(\eta, k)$. Fig. 5 shows an example of the generalized windmill graph of Type III.

In this paper we will study properties for all three types of generalized windmill graphs. In particular we will focus on the behavior of the two clustering metrics. We will quantify the difference between the two metrics, under various conditions. In addition, we give the spectra of the adjacency and the Laplacian matrices of these graphs. We will also derive analytic expressions for several other graph metrics, such as average path length, heterogeneity index and a variety of robustness metrics. We also show how the generalized windmill graphs can be used to construct pairs of non-isomorphic graphs with the same number of nodes and links. Finally, we will show how generalized windmill graphs occur quite naturally in the study of public transportation networks.

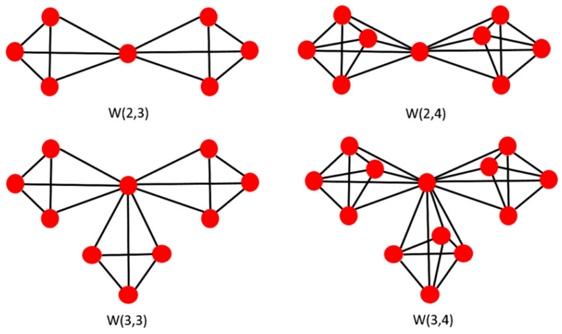



**Fig. 1.** Some small windmill graphs $W(\eta, k)$.

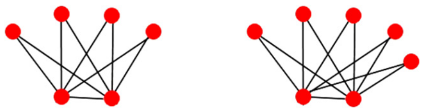


**Fig. 2.** Illustration of agave graphs with 6 nodes and 7 nodes.

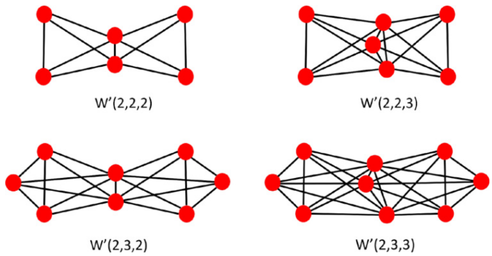



**Fig. 3.** Some generalized windmill graphs $W'(\eta, k, l)$.

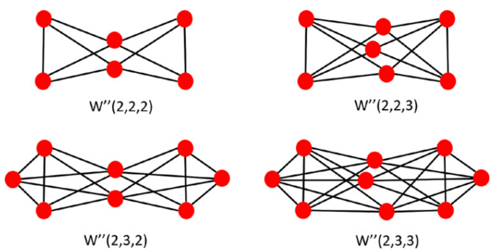



**Fig. 4.** Some generalized windmill graphs $W''(\eta, k, l)$.

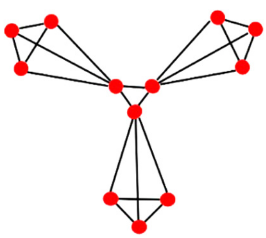


**Fig. 5.** Example of a generalized windmill graph of Type III: $W'''(3, 3)$.

## 2. Generalized windmill graphs of Type I

### *2.1. Clustering metrics*

In this section we will study the clustering coefficient and the transitivity index for generalized windmill graphs of Type I. The clustering coefficient $\bar{C}$ is defined as the local clustering $C_i$, averaged over all nodes, see [18]. Here

$$C_i = \frac{2t_i}{d_i(d_i - 1)}, \tag{1}$$

where $t_i$ is the number of triangles containing the node $i$ and $d_i$ is the degree of node $i$. Denoting the number of nodes by $N$, the clustering coefficient is thus given by

$$\bar{C} = \frac{1}{N}\sum_{i=1}^{N} C_i. \tag{2}$$

The transitivity index $C$, see [13], is defined as

$$C = \frac{3T}{P_2}, \tag{3}$$

where $T$ denotes the total number of triangles and $P_2$ the total number of two-paths, i.e. $P_2 = \sum_{i=1}^{N} \frac{d_i(d_i-1)}{2}$. The clustering coefficient reflects local clustering properties while the transitivity index captures global clustering.

We will prove that the clustering coefficient and the transitivity index diverge when the number of nodes increases in a generalized windmill graph of Type I.

**Theorem 1.** *Let $W'(\eta, k, l)$ be a generalized windmill graph op Type I, with $k > 1$ or $l > 1$. Then, for given values of $k$ and $l$, the clustering coefficient $\bar{C}$ and the transitivity index $C$ diverge when the number of cliques tends to infinity:*

$$\lim_{\eta\to\infty} \bar{C} = 1, \tag{4}$$

$$\lim_{\eta\to\infty} C = 0. \tag{5}$$

**Proof.** First, we obtain an expression for the clustering coefficient of generalized windmill graphs of Type I. The clustering coefficient of each node in one of the $\eta$ copies of the complete graph $K_k$, satisfies $C_j = 1$, because we excluded the case $k = 1 \wedge l = 1$. Next we determine the clustering coefficient for each of the $l$ central nodes. The number of neighbors for each of these nodes is $\eta k + l - 1$. Therefore, the maximum number of connections between these neighbors is $\binom{\eta k+l-1}{2}$. Because in $W'(\eta, k, l)$ the connections of nodes belonging to different pairs of the $\eta$ cliques $K_k$, equals $\binom{\eta}{2}k^2$, the total number of connections between the neighbors of each central node satisfies $\binom{\eta k+l-1}{2} - \binom{\eta}{2}k^2$. Thus, the clustering coefficient $C_i$ for each of the $l$ central nodes is given by $C_i = \frac{\binom{\eta k+l-1}{2} - \binom{\eta}{2}k^2}{\binom{\eta k+l-1}{2}}$. Thus, $\bar{C} = \frac{\eta k + l C_i}{\eta k + l}$, which gives

$$\bar{C} = 1 - \frac{lk^2\eta(\eta-1)}{(\eta k+l)(\eta k+l-1)(\eta k+l-2)}. \tag{6}$$

Now we consider the transitivity index of a generalized windmill graph of Type I $W'(\eta, k, l)$. The total number of triangles $T$ in $W'(\eta, k, l)$ satisfies

$$T = \eta\binom{k+l}{3} - (\eta-1)\binom{l}{3}. \tag{7}$$

This can be shown as follows. First consider one of the $\eta$ cliques $K_k$. This clique forms a larger clique with the $l$ central nodes, of size $k+l$. Therefore the number of triangles in this larger clique equals $\binom{k+l}{3}$. To get all the triangles we have to multiply this number with $\eta$ but subtract $\eta - 1$ times the number of triangles in the clique formed by the $l$ central nodes, given by $\binom{l}{3}$. This leads to Eq. (7).

Next we determine $P_2$, the number of two-paths, for generalized windmill graph of Type I $W'(\eta, k, l)$. For each node in one of the $\eta$ copies of the complete graph $K_k$, it holds that $d_j = k+l-1$. Similarly for the $l$ central nodes the degree is given by $d_i = \eta k + l - 1$. It follows that $P_2 = \eta k \frac{d_j(d_j-1)}{2} + l\frac{d_i(d_i-1)}{2}$, therefore

$$P_2 = \eta k\binom{k+l-1}{2} + l\binom{\eta k+l-1}{2}. \tag{8}$$

Combining Eqs. (7) and (8) gives

$$C = \frac{3\eta\binom{k+l}{3} - 3(\eta-1)\binom{l}{3}}{\eta k\binom{k+l-1}{2} + l\binom{\eta k+l-1}{2}}. \tag{9}$$

Obviously, for given values of $k$ and $l$, $lim_{\eta\to\infty}\bar{C} = 1$ and $lim_{\eta\to\infty}C = 0$, which proves the theorem. For the excluded case $k = 1 \wedge l = 1$, both $\bar{C}$ and $C$ are zero.

Theorem 1 holds under the condition that $k$ and $l$ are fixed, while $\eta$ becomes unbounded. We will now show that the behavior of the metrics $\bar{C}$ and $C$ is different when the number of central nodes $l$ scales linearly with $\eta$.

**Theorem 2.** *Let $W'(\eta, k, l)$ be a generalized windmill graph op Type I. Assume $l$ scales linearly with $\eta$, i.e. $l = a\eta$. Then, for given values of $k$ and $a$, the clustering coefficient $\bar{C}$ and the transitivity index $C$ diverge when the number of cliques tends to infinity:*

$$\lim_{\eta\to\infty}\bar{C} = 1 - \frac{\mu}{(\mu+1)^3} = \bar{C}_\infty(\mu), \tag{10}$$

$$\lim_{\eta\to\infty} C = 1 - \frac{1}{(\mu+1)^2+\mu} = C_\infty(\mu), \tag{11}$$

*where* $\mu = \frac{a}{k}$.

**Proof.** Upon substitution of $l = a\eta$, we see that the leading term in the numerator of the second term in the right hand side of Eq. (6) becomes $ak^2\eta^3$. Similarly, the leading term in the denominator of the second term in the right hand side of Eq. (6) becomes $(a+k)^3\eta^3$. Putting $a = \mu k$, and taking the limit for $\eta \to \infty$ we obtain Eq. (10).

Next, putting $l = a\eta$, in Eq. (7) and writing out the whole expression, we get

$$T = \frac{(3k+a)a^2}{6}\eta^3 + O(\eta^2). \tag{12}$$

Similarly, we can show that for the number of two-paths $P_2$, see Eq. (8), it holds

$$P_2 = \frac{(a^2k + a(a+k)^2)}{2}\eta^3 + O(\eta^2). \tag{13}$$

Substitution of Eqs. (12) and (13) into $C = \frac{3T}{P_2}$, putting $a = \mu k$ and taking the limit for $\eta \to \infty$, we obtain Eq. (11).

It can be shown that for any value $b > 1$, there exists a unique value $\mu > 0$ such that $\bar{C}_\infty(\mu) = bC_\infty(\mu)$. To see this, we use Eqs. (10) and (11) and substitute them into $\bar{C}_\infty(\mu) = bC_\infty(\mu)$, yielding:

$$(b-1)\mu^5 + 6(b-1)\mu^4 + 12(b-1)\mu^3 + 10(b-1)\mu^2 + (3b-5)\mu - 1 = 0. \tag{14}$$

Applying Descartes' rule of signs, we see that for $b > 1$ Eq. (14) has exactly one positive root.

As an example, consider the case $b = 2$. Then Eq. (14) becomes

$$\mu^5 + 6\mu^4 + 12\mu^3 + 10\mu^2 + \mu - 1 = 0. \tag{15}$$

Let us denote the unique positive root of Eq. (15) by $\mu_2$. Then indeed, it holds that $\bar{C}_\infty(\mu_2) = 2C_\infty(\mu_2)$. To find the corresponding generalized windmill graph, we need to approximate $\mu_2$ by an appropriate quotient $a/k$, where both $a$ and $k$ are integers. Solving Eq. (15) numerically we find $\mu_2 \approx 0.2396$.

Table 1 shows the results for the actual values of $\bar{C}$ and $C$, for $k = 10$ and $\eta = 10000$, when we take an increasing number of digits into account from $\mu_2$, i.e. we consider the following values for $\mu_2 : \{0.2, 0.23, 0.239, 0.2396\}$. Note that because $\mu = a/k$ and $l = a\eta$ we have $l = k\mu\eta$.

### 2.2. Other metrics

Apart from the clustering metrics discussed above, the special structure of the windmill graphs also allows explicit expressions for a variety of other graph metrics. In

**Table 1**
Generalized windmill graphs of Type I with ratio of $\bar{C}$ and $C$ approaching 2.

| $\eta$ | $k$ | $l$ | $\bar{C}$ | $C$ | $\bar{C}/C$ |
|---|---|---|---|---|---|
| 10000 | 10 | 20000 | 0.88427 | 0.39035 | 2.26529 |
| 10000 | 10 | 23000 | 0.87641 | 0.42634 | 2.05564 |
| 10000 | 10 | 23900 | 0.87435 | 0.43644 | 2.00338 |
| 10000 | 10 | 23960 | 0.87422 | 0.43710 | 2.00004 |

particular we will derive expressions for the number of nodes and edges, the degree heterogeneity and the average path length.

The number of nodes $N$ is the sum of all nodes in the $\eta$ cliques and the $l$ central nodes, leading to

$$N = \eta k + l. \tag{16}$$

The number of links $L$ is determined by the links in each of the $\eta$ cliques ($\binom{k}{2}$ per clique), the links connecting central nodes to the cliques ($k$ links per central node for each clique) and the links within the core clique ($\binom{l}{2}$ per clique). This leads to

$$L = \eta \left( \binom{k}{2} + kl \right) + \binom{l}{2}. \tag{17}$$

From Eqs. (16) and (17) we can also determine the average degree $D$, satisfying $D = 2L/N$.

Next we derive an expression for the heterogeneity index $H$, see [3] defined as

$$H = \frac{1}{N} \sum_{j=1}^{N} (d_i - D)^2. \tag{18}$$

**Theorem 3.** *Let $W'(\eta, k, l)$ be a generalized windmill graph op Type I. Then, the heterogeneity index $H$ is given by*

$$H = \frac{\eta(\eta-1)^2 k^3 l}{(\eta k + l)^2}. \tag{19}$$

**Proof.** For each node $n_j$ in one of the $\eta$ copies of the complete graph $K_k$ it holds that $d_j = k + l - 1$. For each central node $n_i$ it holds that $d_i = \eta k + l - 1$. Writing out $H = \frac{\eta k (d_j - D)^2 + l(d_i - D)^2}{\eta k + l}$ leads to Eq. (19).

**Theorem 4.** *Let $W'(\eta, k, l)$ be a generalized windmill graph op Type I. Then, the average path length $\bar{l}$ is given by*

$$\bar{l} = \frac{\eta k(2\eta k - k - 1) + l(2\eta k + l - 1)}{(\eta k + l)(\eta k + l - 1)}. \tag{20}$$

**Proof.** For each node $n_j$ in one of the $\eta$ copies of the complete graph $K_k$ it holds that it is part of a clique of size $k+l$, therefore, the node contributes $k+l-1$ shortest paths of length 1. The remaining $(\eta-1)k$ nodes outside this clique are at distance 2. Therefore the total shortest path lengths from node $n_j$ equals $l_j = k+l-1+2(\eta-1)k$. Each central node $n_i$ is connected to all other $\eta\kappa + l - 1$ nodes. Therefore the total shortest path lengths from node $n_i$ equals $l_i = \eta\kappa + l - 1$. It follows that the total shortest paths lengths from all nodes equal

$$(\eta k(k+l-1+2(\eta-1)k) + l(\eta k + l - 1))/2 \tag{21}$$

Note that we divided by a factor 2 because in our construction, every shortest path is counted twice. Finally, dividing Eq. (21) by the total number of node pairs, i.e. $\binom{\eta k+l}{2}$, we obtain Eq. (20).

The next two theorems are similar to Theorems 1 and 2, but now for the behavior of the average path length $\bar{l}$, as the number of cliques tends to infinity. The proofs are similar as for the aforementioned theorems and are omitted here.

**Theorem 5.** *Let $W'(\eta,k,l)$ be a generalized windmill graph op Type I. Then, for given values of $k$ and $l$, then the average path length $\bar{l}$ tends to 2 when the number of cliques tends to infinity.*

**Theorem 6.** *Let $W'(\eta,k,l)$ be a generalized windmill graph op Type I. Assume $l$ scales linearly with $\eta$, i.e. $l = a\eta$. Then, for given values of $k$ and $a$, when the number of cliques tends to infinity, then the average path length $\bar{l}$ tends to a value smaller than 2:*

$$\lim_{\eta\to\infty} \bar{l} = 1 + \frac{1}{(\mu+1)^2}, \tag{22}$$

*where $\mu = \frac{a}{k}$.*

With the help of Eq. (22) we can construct generalized windmill graphs of Type I with a given average path length, arbitrary close to any number between 1 and 2. As an example, we consider, $\lim_{\eta\to\infty} \bar{l} = \frac{5}{4}$. Obviously, this case corresponds with $\mu = 1$, i.e. $a = k$. For example, the graph $W'(10000, 3, 30000)$ has average path length $\bar{l} = 1.249979$.

### *2.3. Spectral metrics*

In this section we determine the spectrum of both the adjacency matrix and the Laplacian matrix of the generalized windmill graph of Type I.

As a starting point we will give the adjacency matrix for $W'(\eta,k,l)$ and denote it by $A(W'(\eta,k,l))$.

$$A(W'(\eta,k,l)) = \begin{bmatrix} (J-I)_{l\times l} & J_{l\times k} & J_{l\times k} & \dots & J_{l\times k} \\ J_{k\times 1} & (J-I)_{k\times k} & 0_{k\times k} & \dots & 0_{k\times k} \\ J_{k\times 1} & 0_{k\times k} & (J-I)_{k\times k} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ J_{k\times 1} & 0_{k\times k} & \dots & \dots & (J-I)_{k\times k} \end{bmatrix}, \tag{23}$$

where $J$ denotes the all-ones matrix. The degree distribution for $W'(\eta,k,l)$ is bi-modal: the $l$ core nodes all have degree $d_{core} = \eta k + l - 1$, while the $\eta k$ nodes in the $\eta$ cliques have degree $d_{clique} = k - 1 + l$.

From this it follows that the degree matrix $\Delta$ satisfies

$$\Delta = \begin{bmatrix} d_{core} I_{l\times l} & 0_{l\times \eta k} \\ 0_{\eta k\times l} & d_{clique} I_{\eta k\times \eta k} \end{bmatrix}. \tag{24}$$

Combining this with Eq. (23), we obtain the expression for the Laplacian matrix $Q(W'(\eta,k,l))$:

$$Q(W'(\eta,k,l)) = \begin{bmatrix} (rI-J)_{l\times l} & -J_{l\times k} & -J_{l\times k} & \dots & -J_{l\times k} \\ -J_{k\times 1} & (tI-J)_{k\times k} & 0_{k\times k} & \dots & 0_{k\times k} \\ -J_{k\times 1} & 0_{k\times k} & (tI-J)_{k\times k} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ -J_{k\times 1} & 0_{k\times k} & \dots & \dots & (tI-J)_{k\times k} \end{bmatrix}, \tag{25}$$

with $r = d_{core} + 1 = \eta k + l$ and $t = d_{clique} = k + l$.

**Theorem 7.** *The spectrum of the adjacency matrix of the generalized windmill graph of Type I $W'(\eta,k,l)$ is*

$$\{(-1)^{\eta(k-1)+l-1}, (k-1)^{\eta-1}, {\lambda_1}^1, {\lambda_2}^1\}, \tag{26}$$

*where $\lambda_1$ and $\lambda_2$ satisfy $\lambda^2 - (k+l-2)\lambda + (k-1)(l-1) - \eta kl = 0$.*

**Proof.** We first consider the $\eta k + l$-dimensional vector $v_1 = [1,\dots,1,x,\dots,x]^T$, where the first $l$ entries are one and $x$ will be determined later. Then it follows that

$$Av_1 = [l-1+\eta kx,\dots,l-1+\eta kx, l+(k-1)x,\dots,l+(k-1)x]^T. \tag{27}$$

Assuming that the right hand side of Eq. (27) equals $\lambda v_1$, we obtain

$$l - 1 + \eta kx = \lambda, \tag{28}$$

$$l + (k-1)x = \lambda x, \tag{29}$$

which leads to

$$\lambda^2 - (k+l-2)\lambda + (k-1)(l-1) - \eta kl = 0, \tag{30}$$

which leads to two eigenvalues $\lambda_1$ and $\lambda_2$, each with multiplicity one. The constant $x$ satisfies $x = \frac{\lambda - l + 1}{\eta k}$.

Next, let $v_2 = [\beta_1, \ldots, \beta_l, 0, \ldots, 0]^T$, be a $\eta k + l$-dimensional vector such that $\sum_{j=1}^{l} \beta_j = 0$ and $\beta_j \neq 0$ for some $j$. Then, as a result $Av_2 = -v_2$. Similarly, let $v_3 = [0, \ldots, 0, \alpha_{11}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots, \alpha_{\eta 1}, \ldots, \alpha_{\eta k}]^T$, be a $\eta k + l$-dimensional vector where the first $l$ entries are zero, such that $\sum_{j=1}^{k} \alpha_{mj} = 0$ for all $m \in \{1, \ldots, \eta\}$ and $\alpha_{mj} \neq 0$ for some $m$ and $j$. It also holds that $Av_3 = -v_3$. Therefore, there exists a set of $\eta(k-1) + l - 1$ orthogonal eigenvectors $\{v_2, v_3\}$, implying that $-1$ is an eigenvalue of $A(W'(\eta, k, l))$ with multiplicity $\eta(k-1) + l - 1$.

Finally, let $v_4 = [0, \ldots, 0, \alpha_{11}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots, \alpha_{\eta 1}, \ldots, \alpha_{\eta k}]^T$, where the first $l$ entries are zero, such that for all $m \in \{1, \ldots, \eta\}$ it holds $\alpha_{mj} = \alpha_m$, $\sum_{j=1}^{\eta} \alpha_j = 0$ and $\alpha_j \neq 0$ for some $j$. Then $Av_4 = (k-1)v_4$. Therefore, there exists a set of $\eta - 1$ orthogonal eigenvectors $\{v_4\}$, implying that $k - 1$ is an eigenvalue of $A(W'(\eta, k, l))$ with multiplicity $\eta - 1$.

Because the sum of the multiplicities of the found eigenvalues equals the number of nodes $\eta k + l$, we have found all eigenvalues. This finishes the proof.

**Remark.** Theorem 7 can be used as an alternative way to derive an expression for the number of triangles $T$, see Eq. (7), through the identity $T = \frac{1}{6} \sum_{j=1}^{N} \lambda_i^3$, see [15].

**Theorem 8.** *The spectrum of the Laplacian matrix of the generalized windmill graph of Type I $W'(\eta, k, l)$ is*

$$\{0^1, l^{\eta-1}, (k+l)^{\eta(k-1)}, (\eta k + l)^l\}. \tag{31}$$

**Proof.** It is a well-known fact that the Laplacian matrix has a zero eigenvalue, with the all-ones vector as eigenvector. Because the generalized windmill graph is connected, its multiplicity is one. Next we let $v_5 = [0, \ldots, 0, \alpha_{11}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots, \alpha_{\eta 1}, \ldots, \alpha_{\eta k}]^T$, be a $\eta k + l$-dimensional vector such that $\sum_{j=1}^{k} \alpha_{mj} = 0$ for all $m \in \{1, \ldots, \eta\}$ and $\alpha_{mj} \neq 0$ for some $m$ and $j$. Then, as a result, $Qv_5 = (k + l)v_5$. Therefore, there exists a set of $\eta(k-1)$ orthogonal eigenvectors $\{v_5\}$, implying that $k + l$ is an eigenvalue of $Q(W'(\eta, k, l))$ with multiplicity $\eta(k-1)$.

As in the proof of Theorem 7 we again consider the vector $v_4$. Then it follows that $Qv_4 = lv_4$. Therefore, there exists a set of $\eta - 1$ orthogonal eigenvectors $\{v_4\}$, implying that $l$ is an eigenvalue of $Q(W'(\eta, k, l))$ with multiplicity $\eta - 1$.

Next, consider $v_6 = [1, \ldots, 1, -\frac{l}{\eta k}, \ldots, -\frac{l}{\eta k}]^T$. Then it can be shown that $Qv_6 = (\eta k + l)v_6$. Finally, let $v_7 = [\beta_1, \ldots, \beta_l, 0, \ldots, 0]^T$, with $\sum_{j=1}^{l} \beta_j = 0$ and $\beta_j \neq 0$ for some $j$. It also holds that $Qv_7 = (\eta k + l)v_7$. Therefore, there exists a set of $l$ orthogonal eigenvectors $\{v_6, v_7\}$, implying that $\eta k + l$ is an eigenvalue of $Q(W'(\eta, k, l))$ with multiplicity $l$.

Because the sum of the multiplicities of the found Laplacian eigenvalues equals the number of nodes $\eta k + l$, we have found all Laplacian eigenvalues. This finishes the proof.

**Remark.** Theorem 8 can be used as an alternative way to derive an expression for the number of two-paths $P_2$, see Eq. (8), through the identity $\sum_{j=1}^{N} \mu_i^2 = 2P_2 - 4L$, see [15].

We end this section with an example, the generalized windmill graph of Type I: $W'(2,3,2)$, which is also depicted in Fig. 3.

According to Theorem 7, the adjacency eigenvalues are 5, 2, −1 and −2. Using the proof of Theorem 7 it is easy to derive a set of orthogonal eigenvectors: $\lambda = 5$: $[1, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}]^T$, $\lambda = -2$: $[1, 1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]^T$, $\lambda = -1 : \{[1, -1, 0, 0, 0, 0, 0, 0]^T, [0, 0, 1, -1, 0, 0, 0, 0]^T, [0, 0, 1, 1, -2, 0, 0, 0]^T, [0, 0, 0, 0, 0, 1, -1, 0]^T, [0, 0, 0, 0, 0, 1, 1, -2]^T\}$, $\lambda = 2 : [0, 0, 1, 1, 1, -1, -1, -1]^T$.

The Laplacian eigenvalues are 8, 5, 2 and 0, applying Theorem 8. A set of orthogonal eigenvectors is as follows: $\mu = 8 : \{[1, 1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}]^T, [1, -1, 0, 0, 0, 0, 0, 0]^T\}$, $\mu = 5 : \{[0, 0, 1, -1, 0, 0, 0, 0]^T, [0, 0, 1, 1, -2, 0, 0, 0]^T, [0, 0, 0, 0, 0, 1, -1, 0]^T, [0, 0, 0, 0, 0, 1, 1, -2]^T\}$, $\mu = 2 : [0, 0, 1, 1, 1, -1, -1, -1]^T$, $\mu = 0 : [1, 1, 1, 1, 1, 1, 1, 1]^T$.

## 3. Generalized windmill graphs of Type II

In this section we consider generalized windmill graphs of Type II. Here we will just state the results. The proofs are similar to the ones in the previous section and are therefore omitted.

### 3.1. Clustering metrics

**Theorem 9.** *Let $W''(\eta, k, l)$ be a generalized windmill graph of Type II, with $\eta > 1$ or $k > 1$. Then, the clustering coefficient $\bar{C}$ and the transitivity index $C$ satisfy:*

$$\bar{C} = 1 - \frac{l}{\eta k + l} - \frac{\eta k l(l-1)}{(\eta k + l)(k + l - 1)(k + l - 2)} + \frac{l(k-1)}{(\eta k + l)(\eta k - 1)}, \tag{32}$$

$$C = \frac{(k-1)(3l + k - 2)}{\eta k l + l^2 + 2kl - 4l + k^2 - 3k + 2}. \tag{33}$$

*For given values of $k$ and $l$, the clustering coefficient $\bar{C}$ and the transitivity index $C$ diverge, when the number of cliques tends to infinity:*

$$\lim_{\eta \to \infty} \bar{C} = 1 - \frac{l(l-1)}{(k + l - 1)(k + l - 2)}, \tag{34}$$

$$\lim_{\eta \to \infty} C = 0. \tag{35}$$

For the excluded case $\eta = 1 \wedge k = 1$ both $\bar{C}$ and $C$ equal zero.

**Remark.** In the derivation of Theorem 9 we have established that the number of triangles $T$ in $W''(\eta, k, l)$ satisfies $T = \eta \left( \binom{k}{3} + l \binom{k}{2} \right)$, while $P_2 = \eta k \binom{k+l-1}{2} + l \binom{\eta k}{2}$.

*3.2. Other metrics*

The number of nodes $N$ and number of links $L$ for the generalized windmill graph of Type II $W''(\eta, k, l)$, are given by

$$N = \eta k + l, \tag{36}$$

$$L = \eta \left( \binom{k}{2} + kl \right). \tag{37}$$

**Theorem 10.** *Let $W''(\eta, k, l)$ be a generalized windmill graph op Type II. Then, the heterogeneity index $H$ is given by*

$$H = \frac{\eta kl(\eta k - k - l + 1)^2}{(\eta k + l)^2}. \tag{38}$$

**Theorem 11.** *Let $W''(\eta, k, l)$ be a generalized windmill graph op Type II. Then, the average path length $\bar{l}$ is given by*

$$\bar{l} = \frac{\eta k(2\eta k - k - 1) + 2l(\eta k + l - 1)}{(\eta k + l)(\eta k + l - 1)}. \tag{39}$$

*3.3. Spectral metrics*

**Theorem 12.** *The spectrum of the adjacency matrix of the generalized windmill graph of Type II $W''(\eta, k, l)$ is*

$$\{0^{l-1}, (-1)^{\eta(k-1)}, (k-1)^{\eta-1}, {\lambda_1}^1, {\lambda_2}^1\}, \tag{40}$$

*where $\lambda_1$ and $\lambda_2$ satisfy $\lambda^2 - (k-1)\lambda - \eta kl = 0$.*

**Remark.** The corresponding eigenvectors are as follows: for $\lambda = 0$: $[\beta_1, \dots, \beta_l, 0, \dots, 0]^T$, with $\sum_{j=1}^{l} \beta_j = 0$ and $\beta_j \neq 0$ for some $j$; for $\lambda = -1$: $[0, \dots, 0, \alpha_{11}, \dots, \alpha_{1k}, \alpha_{21}, \dots, \alpha_{2k}, \dots, \alpha_{\eta 1}, \dots, \alpha_{\eta k}]^T$, where the first $l$ entries are zero and $\sum_{j=1}^{k} \alpha_{mj} = 0$ for all $m \in \{1, \dots, \eta\}$ and $\alpha_{mj} \neq 0$ for some $m$ and $j$; for $\lambda = k - 1$: $[0, \dots, 0, \alpha_{11}, \dots, \alpha_{1k}, \alpha_{21}, \dots, \alpha_{2k}, \dots, \alpha_{\eta 1}, \dots, \alpha_{\eta k}]^T$, where the first $l$ entries are zero, such that for all $m \in \{1, \dots, \eta\}$ it holds $\alpha_{mj} = \alpha_m$, $\sum_{j=1}^{\eta} \alpha_j = 0$ and $\alpha_j \neq 0$ for some $j$, for $\lambda = \lambda_1$ and $\lambda_2$: $[1, \dots, 1, x, \dots, x]^T$, where the first $l$ entries are one and $x = \frac{\lambda}{\eta k}$.

**Theorem 13.** *The spectrum of the Laplacian matrix of the generalized windmill graph of Type II $W''(\eta, k, l)$ is*

$$\{0^1, l^{\eta-1}, (k+l)^{\eta(k-1)}, (\eta k)^{l-1}, (\eta k + l)^1\}. \tag{41}$$

**Remark.** The corresponding eigenvectors are as follows: for $\mu = 0$: $[1, \ldots, 1]^T$, the all-ones vector; for $\mu = l$: $[0, \ldots, 0, \alpha_{11}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots, \alpha_{\eta 1}, \ldots, \alpha_{\eta k}]^T$, where the first $l$ entries are zero, such that for all $m \in \{1, \ldots, \eta\}$ it holds $\alpha_{mj} = \alpha_m$, $\sum_{j=1}^{\eta} \alpha_j = 0$ and $\alpha_j \neq 0$ for some $j$, for $\mu = k + l$: $[0, \ldots, 0, \alpha_{11}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots, \alpha_{\eta 1}, \ldots, \alpha_{\eta k}]^T$, where the first $l$ entries are zero and $\sum_{j=1}^{k} \alpha_{mj} = 0$ for all $m \in \{1, \ldots, \eta\}$ and $\alpha_{mj} \neq 0$ for some $m$ and $j$; for $\mu = \eta k$: $[\beta_1, \ldots, \beta_l, 0, \ldots, 0]^T$, with $\sum_{j=1}^{l} \beta_j = 0$ and $\beta_j \neq 0$ for some $j$; for $\mu = \eta k + l$: $[1, \ldots, 1, -\frac{l}{\eta k}, \ldots, -\frac{l}{\eta k}]^T$.

## 4. Generalized windmill graphs of Type III

In this section we consider generalized windmill graphs of Type III. Again, we will only state the results, without proofs.

### *4.1. Clustering metrics*

**Theorem 14.** *Let $W'''(\eta, k)$ be a generalized windmill graph of Type III, with $k > 1$. Then, the clustering coefficient $\bar{C}$ and the transitivity index $C$ satisfy:*

$$\bar{C} = 1 - \frac{2k(\eta - 1)}{(k+1)(\eta + k - 1)(\eta + k - 2)}. \tag{42}$$

$$C = \frac{\eta^2 - 3\eta + k^3 - k + 2}{\eta^2 - 3\eta + k^3 - k + 2 + 2k(\eta - 1)} \tag{43}$$

*For a given value of $k > 1$ the clustering coefficient $\bar{C}$ and the transitivity index $C$ converge to 1 when the number of cliques tends to infinity. For $k = 1$ the clustering coefficient satisfies $\bar{C} = \frac{\eta - 2}{2\eta}$, while Eq. (43) still holds for $k = 1$.*

**Remark.** In the derivation of Theorem 14 we have established that the number of triangles $T$ in $W'''(\eta, k)$ satisfies $T = \eta\binom{k+1}{3} + \binom{\eta}{3}$, while $P_2 = \frac{\eta k^2(k-1)}{2} + \frac{\eta(\eta+k-1)(\eta+k-2)}{2}$.

### *4.2. Other metrics*

The number of nodes $N$ and number of links $L$ for the generalized windmill graph of Type III $W'''(\eta, k)$, are given by

$$N = \eta k + \eta \tag{44}$$

$$L = \eta\binom{k+1}{2} + \binom{\eta}{2}. \tag{45}$$

**Theorem 15.** *Let $W'''(\eta, k)$ be a generalized windmill graph op Type III. Then, the heterogeneity index $H$ is given by*

$$H = \frac{k(\eta - 1)^2}{k + 1^2}. \tag{46}$$

**Theorem 16.** *Let $W'''(\eta, k)$ be a generalized windmill graph op Type III. Then, the average path length $\bar{l}$ is given by*

$$\bar{l} = \frac{\eta(k + 1)(3\eta k + \eta - 2k - 1)}{(\eta k + \eta)(\eta k + \eta - 1)}. \tag{47}$$

*4.3. Spectral metrics*

**Theorem 17.** *The spectrum of the adjacency matrix of the generalized windmill graph of Type III $W'''(\eta, k)$ is*

$$\{(-1)^{\eta(k-1)}, {\lambda_1}^1, {\lambda_2}^1, {\lambda_3}^{\eta-1}, {\lambda_4}^{\eta-1}\}, \tag{48}$$

*where $\lambda_1$ and $\lambda_2$ satisfy $\lambda^2 - (\eta + k - 2)\lambda + 1 - 2k - \eta + \eta k$ and $\lambda_3$ and $\lambda_4$ satisfy $\lambda^2 - (k - 2)\lambda + 1 - 2k$.*

**Remark.** The corresponding eigenvectors are as follows: for $\lambda = -1$: $[0, \ldots, 0, \alpha_{11}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots, \alpha_{\eta 1}, \ldots, \alpha_{\eta k}]^T$, where the first $\eta$ entries are zero and $\sum_{j=1}^{k} \alpha_{mj} = 0$ for all $m \in \{1, \ldots, \eta\}$ and $\alpha_{mj} \neq 0$ for some $m$ and $j$; for $\lambda = \lambda_1$ and $\lambda_2$: $[1, \ldots, 1, x, \ldots, x]^T$, where the first $\eta$ entries are one and $x = \frac{\lambda - \eta + 1}{k}$; for $\lambda = \lambda_3$ and $\lambda_4$; $[\beta_1, \ldots, \beta_\eta, x\beta_1, \ldots, x\beta_1, x\beta_2, \ldots, x\beta_2, \ldots, x\beta_\eta, \ldots, x\beta_\eta]^T$, with $\sum_{j=1}^{l} \beta_j = 0$ and $\beta_j \neq 0$ for some $j$ and $x = \frac{\lambda + 1}{k}$.

**Theorem 18.** *The spectrum of the Laplacian matrix of the generalized windmill graph of Type III $W'''(\eta, k)$ is*

$$\{0^1, (k + l)^{\eta(k-1)+1}, {\mu_1}^{\eta-1}, {\mu_2}^{\eta-1}\}, \tag{49}$$

*where $\mu_1$ and $\mu_2$ satisfy $\mu^2 - (\eta + k + 1)\mu + \eta = 0$.*

**Remark.** The corresponding eigenvectors are as follows: for $\mu = 0$: $[1, \ldots, 1]^T$, the all-ones vector; for $\mu = k + 1$: $[0, \ldots, 0, \alpha_{11}, \ldots, \alpha_{1k}, \alpha_{21}, \ldots, \alpha_{2k}, \ldots, \alpha_{\eta 1}, \ldots, \alpha_{\eta k}]^T$, where the first $l$ entries are zero and $\sum_{j=1}^{k} \alpha_{mj} = 0$ for all $m \in \{1, \ldots, \eta\}$ and $\alpha_{mj} \neq 0$ for some $m$ and $j$ and $[1, \ldots, 1, 0, \ldots, -1, 0, \ldots, -1, \ldots, 0, \ldots, -1]^T$; for $\mu = \mu_1$ and $\mu_2$; $[\beta_1, \ldots, \beta_\eta, x\beta_1, \ldots, x\beta_1, x\beta_2, \ldots, x\beta_2, \ldots, x\beta_\eta, \ldots, x\beta_\eta]^T$, with $\sum_{j=1}^{l} \beta_j = 0$ and $\beta_j \neq 0$ for some $j$ and $x = \frac{\eta + k - \mu}{k}$.

## 5. Generalized windmill graphs of the same order and size

In this section we will show that within the classes of generalized windmill graphs we have defined in this paper, it is possible to find pairs of non-isomorphic graphs of the same order and size, i.e. with the same number of nodes and links.

**Theorem 19.** *There exists an infinite number of graph pairs, with one graph a generalized windmill graph of Type I, and the other of Type II, such that the graphs are of the same order and size.*

**Proof.** We will consider $G_1 = W'(\eta_1, k_1, l_1)$ and $G_2 = W''(\eta_2, k_2, l_2)$. Let $N_1$, $L_1$ and $N_2$, $L_2$, denote the number of nodes and links of $G_1$ and $G_2$, respectively.

Then, in order to make $G_1$ and $G_2$ of the same order and size, according to Sections 2 and 3, we have

$$N_1 = \eta_1 k_1 + l_1 = \eta_2 k_2 + l_2 = N_2, \tag{50}$$

$$L_1 = \eta_1 \left( \binom{k_1}{2} + k_1 l_1 \right) + \binom{l_1}{2} = \eta_2 \left( \binom{k_2}{2} + k_2 l_2 \right) = L_2. \tag{51}$$

Under the assumption $l_1 = l_2 = l$, Eqs. (50)–(51) become

$$\eta_1 k_1 = \eta_2 k_2, \tag{52}$$

$$l(l-1) = \eta_1 k_1 (k_2 - k_1). \tag{53}$$

Next we assume $k_2 = a k_1$, with $a$ an integer. Then Eqs. (52)–(53) give

$$\eta_2 = \frac{\eta_1}{a}, \tag{54}$$

$$l(l-1) = (a-1)\eta_1 k_1^2. \tag{55}$$

We now have to find solutions of the Diophantine equation (55). One possible solution is given by

$$l = k_1^2, \tag{56}$$

$$l - 1 = (a-1)\eta_1. \tag{57}$$

As a result, we get

$$\eta_1 = \frac{k_1^2 - 1}{a-1}, \tag{58}$$

$$\eta_2 = \frac{k_1^2 - 1}{a(a-1)}. \tag{59}$$

Finally, in order to make $\eta_1$ and $\eta_2$ integers, we choose

$$k_1 = a(a-1)m + 1, \tag{60}$$

with $m$ also an integer. Hence, for the following parameterization

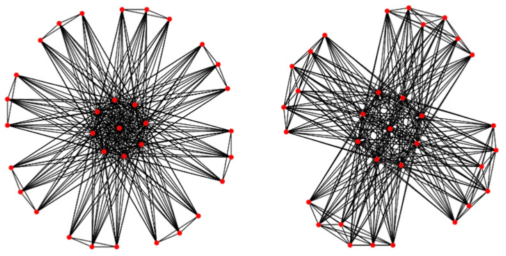



**Fig. 6.** The graphs $W'(8,3,9)$ and $W''(4,6,9)$ with the same order and size.

$$\eta_1 = a^2(a-1)m^2 + 2am, k_1 = a(a-1)m + 1, l_1 = a^2(a-1)^2m^2 + 2a(a-1)m + 1,$$
$$\eta_2 = a(a-1)m^2 + 2m, k_2 = a^2(a-1)m + a, l_2 = a^2(a-1)^2m^2 + 2a(a-1)m + 1, \quad (61)$$

the graphs $W'(\eta_1, k_1, l_1)$ and $W''(\eta_2, k_2, l_2)$ are of the same order and size. This finishes the proof.

As an example we consider the case $a = 2, m = 1$. Then the above parameterization leads to the pair of graphs $W'(8,3,9)$ and $W''(4,6,9)$, which both have $N = 33$ and $L = 276$. A visualization of both graphs is given in Fig. 6.

Other families of examples can be found by further exploring Eq. (55). For instance, the assumption $l = \eta_1$, leads to the parameterization

$$\eta_1 = am, k = (a-1)^b, l_1 = am, \eta_2 = m, k_2 = a(a+1)^b, l_2 = am, \quad (62)$$

where $m = \frac{1+(a-1)(a+1)^{2b}}{a}$, with $b$ an integer.

For example, the case $a = 2, b = 2$ gives the pair of graphs $W'(82,9,82)$ and $W''(41,18,82)$, which both have $N = 820$ and $L = 66789$.

For the examples given above, it holds that $l_1 = l_2$. However, there are also examples with $l_1 \neq l_2$. For instance, $W'(37,4,26)$ and $W''(5,6,144)$ both have $N = 174$ and $L = 4395$.

We end this section with the pair of graphs $W'(6,4,1)$ and $W'''(5,4)$. It is easy to verify that this constitutes an example of a windmill graph and a generalized windmill graph of Type III, of the same order and size, namely $N = 25$ and $L = 60$.

In the next section we will use the obtained results to construct so-called robustness inconsistencies.

## 6. Robustness metrics for generalized windmill graphs

In this section we will give analytic expressions for a number of metrics, that are based upon the adjacency and Laplacian spectrum, that are frequently used in literature to quantify the robustness of graphs [9], [17]. We will consider three metrics based upon the adjacency spectrum and three metrics based upon the Laplacian spectrum.

### 6.1. Robustness metrics

The spectral radius (SR) refers to the largest eigenvalue $\lambda_1$ of the adjacency matrix of a graph: $SR = \lambda_1$. According to the Perron–Frobenius theorem, $\lambda_1$ of a graph is always positive. The smaller the spectral radius is, the higher the robustness of a network with respect to the spread of a virus over the network [16].

The spectral gap (SG) is expressed as $SG = \lambda_1 - \lambda_2$, where $\lambda_2$ denotes the second largest eigenvalue of the adjacency matrix. The higher the spectral gap is, the higher the robustness of a network against link/node removals [19].

Natural connectivity (NC) is defined as $NC = ln\left(1/N \sum_{k=1}^{N} e^{\lambda_k}\right)$, where $\lambda_k$ is the $k$th eigenvalue of the adjacency matrix, see [10]. This metric was first proposed as a robustness metric in [7], where it is shown that natural connectivity can be interpreted as the Helmholtz free energy of a network. The higher the natural connectivity, the higher the robustness of a network.

The algebraic connectivity (AC), refers to the second smallest eigenvalue of the Laplacian matrix $Q$: $AC = \mu_{N-1}$. The larger the algebraic connectivity, the more difficult it is to cut the network into components, hence the higher the robustness of the network [11].

The effective graph resistance (EGR) is determined by $EGR = N\sum_{k=2}^{N} \frac{1}{\mu_k}$, where $\mu_k$ is the $k$th eigenvalue of the Laplacian matrix of a graph. The smaller the effective graph resistance, the higher the robustness of a network [5].

The synchronization ratio refers to the ratio of the second smallest eigenvalue $\mu_{N-1}$ to the largest eigenvalue $\mu_1$ of the Laplacian matrix $Q$ of a graph: $SR = \frac{\mu_{N-1}}{\mu_1}$. The synchronization ratio is used to characterize the synchronizability of networks. The larger the ratio, the better synchronizability the network exhibits [2].

### 6.2. Generalized windmill graphs

In this section we will give analytic expressions for the robustness metrics introduced in the previous section, for generalized windmill graphs.

**Theorem 20.** *The generalized windmill graph of Type I $W'(\eta, k, l)$ has the following spectral robustness metrics: $SR = \lambda_1, SG = \lambda_1 + 1 - k(1 - \delta_{\eta,1})$, $NC = \ln\left(\frac{1}{N}\left((\eta(k-1) + l - 1)e^{-1} + (\eta - 1)e^{k-1} + e^{\lambda_1} + e^{\lambda_2}\right)\right), AC = l$, $EGR = N\left(\frac{\eta-1}{l} + \frac{\eta(k-1)}{k+l} + \frac{l}{N}\right)$ and $SR = \frac{1}{N}$, where $N = \eta k + l$ and $\lambda_1$ and $\lambda_2$ satisfy $\lambda^2 - (k + l - 2)\lambda - (k - 1)(l - 1) - \eta kl = 0$, with $\lambda_1$ the larger root.*

**Proof.** Using Theorems 7–8, it is straightforward to derive expressions for NC and EGR. Also, it is easy to see that the largest Laplacian eigenvalue is $N$, while the second smallest Laplacian eigenvalue is $l$. For $\eta > 1$, the adjacency eigenvalues are $-1, k - 1$ and $\lambda_1$ and $\lambda_2$, the roots of $f(\lambda) = \lambda^2 - (k + l - 2)\lambda - (k - 1)(l - 1) - \eta kl = 0$, with $\lambda_1$ the larger root. It is easy to see that $f(-1) = -kl(\eta - 1) < 0$ and $f(k-1) = -\eta kl < 0$, for $\eta > 1$. Because $f(|x|) > 0$ for sufficiently large $|x|$, it follows that $\lambda_2 \le -1 < k-1 < \lambda_1$, hence $SG = \lambda_1 - k + 1$ for $\eta > 1$. For $\eta = 1$ the second largest eigenvalue is $-1$. With $\delta_{\eta,1}$ denoting the Kronecker delta function, this finishes the proof.

**Theorem 21.** *The generalized windmill graph of Type II $W''(\eta, k, l)$ has the following spectral robustness metrics: $SR = \lambda_1, SG = \lambda_1 - k + 1 + \delta_{\eta,1}(k - \delta_{k,1})$, $NC = \ln\left(\frac{1}{N}\left(\eta(k-1)e^{-1} + (l-1) + (\eta-1)e^{k-1} + e^{\lambda_1} + e^{\lambda_2}\right)\right), AC = \min\{\eta k, l\}$, $EGR = N\left(\frac{\eta-1}{l} + \frac{l-1}{\eta k} + \frac{\eta(k-1)}{k+l} + \frac{1}{N}\right)$ and $SR = \frac{\min\{\eta k, l\}}{N}$, where $N = \eta k + l$ and $\lambda_1$ and $\lambda_2$ satisfy $\lambda^2 - (k-1)\lambda - \eta kl = 0$, with $\lambda_1$ the larger root.*

The proof of the theorem is similar as the proof of Theorem 20 and is therefore omitted.

**Theorem 22.** *The generalized windmill graph of Type III $W'''(\eta, k)$ has the following spectral robustness metrics: $SR = \lambda_1, SG = \lambda_1 - \lambda_3 + \delta_{\eta,1}(\lambda_3 + 1)$, $NC = \ln \frac{1}{N}\left(\eta(k-1)e^{-1} + e^{\lambda_1} + e^{\lambda_2} + (\eta-1)(e^{\lambda_3} + e^{\lambda_4})\right), AC = \mu_2$, $EGR = N\left(\frac{\eta(k-1)+1}{k+1} + \frac{\eta-1}{\mu_1} + \frac{\eta-1}{\mu_2}\right)$ and $SR = \frac{\mu_1}{\mu_2}$, where $N = \eta k + l$ and $\lambda_1$ and $\lambda_2$ satisfy $\lambda^2 - (k+\eta-2)\lambda + 1 - 2k - \eta + \eta k = 0$, with $\lambda_1$ the larger root, $\lambda_3$ and $\lambda_4$ satisfy $\lambda^2 - (k-2)\lambda + 1 - 2k = 0$, with $\lambda_3$ the larger root, $\mu_1$ and $\mu_2$ satisfy $\mu^2 - (\eta + k + 1)\lambda + \eta = 0$, with $\mu_1$ the larger root.*

The proof of Theorem 22 is similar to the previous proofs. Only the proof concerning $SG$ is a bit more elaborate. We only show the case $\eta > 1$. According to Theorem 17 the adjacency eigenvalues are $-1$, $\lambda_1$ and $\lambda_2$ satisfying $g(\lambda) = \lambda^2 - (\eta + k - 2)\lambda + 1 - 2k - \eta + \eta k$ and $\lambda_3$ and $\lambda_4$ satisfying $h(\lambda) = \lambda^2 - (k-2)\lambda + 1 - 2k$. We assume $\lambda_1 > \lambda_2$ and $\lambda_3 > \lambda_4$. Because $h(k-1) = -k < 0$ it follows that

$$k - 1 - \lambda_3 < 0. \tag{63}$$

On the other hand $g(\lambda_3) = \eta(k - 1 - \lambda_3)$, hence applying Eq. (63), we find that $g(\lambda_3) < 0$ which implies that $\lambda_3 < \lambda_1$. This concludes the sketch of the proof.

### *6.3. Inconsistencies among robustness metrics*

In this section we will show the occurrence of inconsistencies among the spectral metrics that quantify robustness. The inconsistencies mean that for a pair of graphs, say $G$ and $H$, the robustness metrics point in opposite direction, i.e. according to one metric $G$ is more robust, but according to the other metric $H$ is more robust. We will give three examples of such inconsistencies for pairs of graphs with the same number of nodes and links.

**Table 2**
Robustness inconsistency for the robustness metrics $NC$ and $EGR$.

| Graph | $N$ | $L$ | $NC$ | $EGR$ |
|---|---|---|---|---|
| $W'(8,3,9)$ | 33 | 276 | 16.50 | 78.67 |
| $W''(4,6,9)$ | 33 | 276 | 13.91 | 67.00 |

**Table 3**
Robustness inconsistency for the robustness metrics $SG$ and $AC$.

| Graph | $N$ | $L$ | $SG$ | $AC$ |
|---|---|---|---|---|
| $W'(37,4,26)$ | 174 | 4395 | 74.00 | 26 |
| $W''(5,6,144)$ | 174 | 4395 | 63.27 | 30 |

**Table 4**
Robustness inconsistency for the robustness metrics $NC$ and $AC$.

| Graph | $N$ | $L$ | $NC$ | $AC$ |
|---|---|---|---|---|
| $W'(6,4,1)$ | 25 | 60 | 3.54 | 1 |
| $W'''(5,4)$ | 25 | 60 | 3.78 | 0.53 |

Non-isomorphic pairs of graphs with the same number of nodes and links were found in Section 5.

Using the results of the previous subsection we can easily construct inconsistencies.

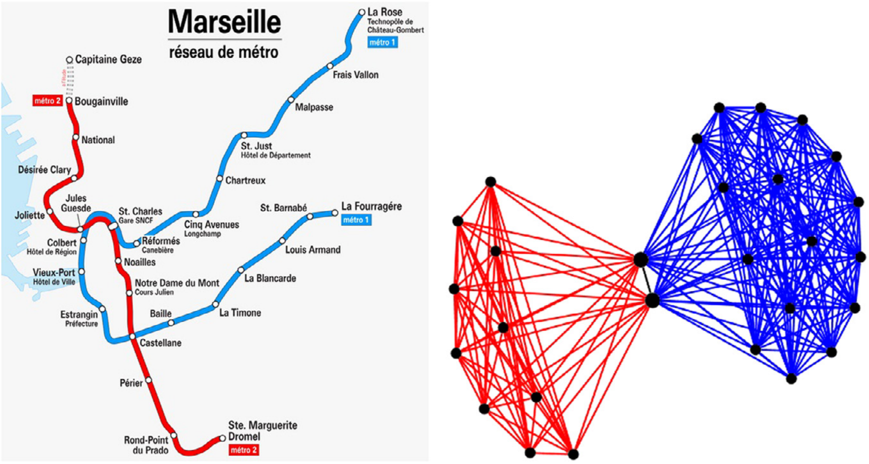
For example, in Table 2 we compare the robustness of the graphs $W'(8,3,9)$ and $W''(4,6,9)$. According to $NC$ $W'(8,3,9)$ is more robust but according to $EGR$ $W''(4,6,9)$ is more robust.

Similar inconsistencies are given in Tables 3 and 4.

## 7. Real-life networks exhibiting a generalized windmill-like structure

It was already shown in [6] that windmill-like structures arise naturally in certain real-world networks. An explicit example is the case of citation networks. It is argued in [6] that seminal papers are often cited by many different groups working in related but different fields. As an example [6] mentions three seminal papers ([12], [18], [1]) and states that these papers are frequently cited by researchers working in mathematics, physics, social sciences, computer sciences and neurosciences. This is illustrated by a citation network of papers citing Milgram's 1967 Psychology Today paper [12] or use "Small World" in the title. This gives rise to a windmill-like structure, with obviously one core node, namely Milgram's paper. Constructing a similar citation network taking more than one, say $l$, seminal paper into account, would naturally lead to a generalized windmill-like structure, with $l$ core nodes.

We will now move our attention to another type of network where also generalized windmill-like structures can be encountered.




(a) Topological representation (b) Representation in $P$-space

**Fig. 7.** Metro network of Marseille. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

An important aspect of the planning of public transport networks (PTNs), is the assessment of the robustness of such networks. In particular one wants to quantify the impact of small disruptions on the network to the overall system performance, expressed for instance as overall travel time reliability, see [14].

According to [4] a PTN can be represented in two different ways. The first way, referred to as $L$-space, represents each station or stop as a node, where two nodes are connected if they represent consecutive stops on at least one service line. In contrast, in so-called $P$-space, nodes still represent stops but now nodes are connected if there is at least one line serving both stops. Cats et al. [4] use $P$-space to model travel time reliability for PTN's and apply this to the urban rail-bound network of Amsterdam.

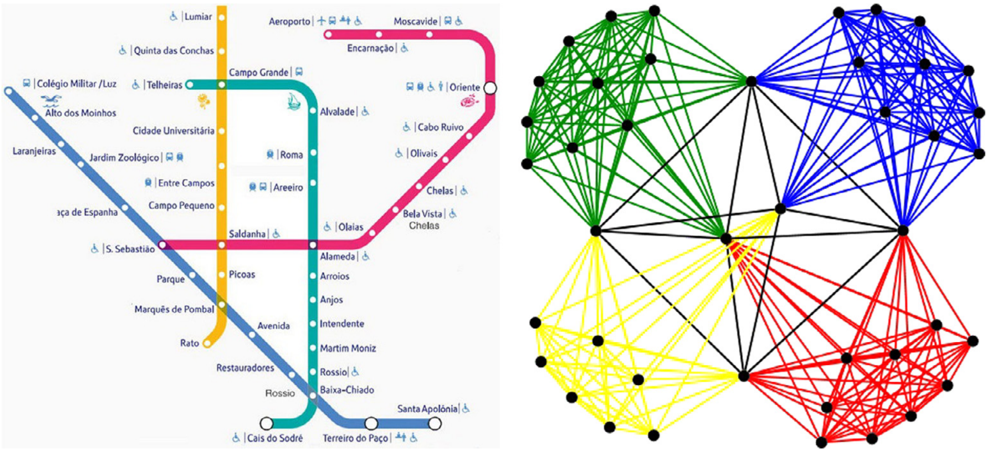
We will now show how generalized windmill-like structures naturally appear when considering PTN's in $P$-space. As a first example we study the metro network of Marseille, see Fig. 7a. Clearly, this PTN consists of two service lines, Metro1 (blue) and Metro2 (red).

Furthermore, the network has two transit stations, servicing both lines, namely St. Charles and Castellane. It is clear that the representation of this metro network in $P$-space gives rise to a structure that resembles a generalized windmill of Type I, with two central nodes, see Fig. 7b.

Note that the network represented in Fig. 7b is not an actual generalized windmill of Type I because the number of nodes in the two cliques representing Metro1 and Metro2 are not equal.

Finally, we look at the metro network of Lisbon, see Fig. 8a. This metro network has 4 service lines, with 6 transit stations.

Again, the representation of this metro network in $P$-space gives rise to a structure that resembles a generalized windmill of Type I, with 6 central nodes, see Fig. 8b.




(a) Topological representation

(b) Representation in $P$-space

**Fig. 8.** Metro network of Lisbon.

## 8. Conclusion

In this paper we have considered generalizations of the class of windmill graphs. Windmill graphs consist of several copies of the complete graph, with every node connected to one central node. Our three generalizations of the windmill graphs all assume that the central node is replaced by several central nodes. The main result of this paper is the fact that it is possible to compute for all three generalized windmill graphs, a variety of graph metrics, such as clustering coefficient and transitivity index, heterogeneity index and average path length, and the spectrum of both the adjacency and Laplacian matrices representing the graphs. As such, this paper generalizes and enhances the results of Estrada [6], who partially studied similar properties in windmill graphs. We have also shown that the generalized windmill graphs can be used to construct pairs of non-isomorphic graphs with the same number of nodes and links. Such pairs can be used to construct so-called robustness inconsistencies. Finally, we have shown how generalized windmill-like structures occur naturally in the study of public transportation networks.

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