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Inflection points for network reliability

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Abstract Given a finite, undirected graph G (possibly with multiple edges), we assume that the vertices are operational, but the edges are each independently operational with probability p. The *(all-terminal) reliability*, $\mathrm{Rel}(G, p)$, of G is the probability that the spanning subgraph of operational edges is connected. It has been conjectured that reliability functions have at most one point of inflection in $(0, 1)$. We show that the all-terminal reliability of almost every simple graph of order n has a point of inflection, and there are indeed infinite families of graphs (both simple and otherwise) with more than one point of inflection.

Keywords Connected · All-terminal reliability · Graph · Point of inflection

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1 Introduction

Our daily activities rely increasingly on large scale technological networks—the power grid, the Internet, transportation systems, to name but a few. These networks are often controlled in a decentralized way and show properties of self-organization. However, even if decentralization and self-organization theoretically reduce the risk of failure, complex networks can experience disruptive and massive failure. In 2001, Code Red, a computer virus that incapacitated numerous networks, resulted in a global loss of 2.6 billion US dollars. In 2004, the Sassar virus caused Delta airlines to cancel 40 transatlantic flights in addition to halting trains in Australia. In another example concerning the power grid, the Northeastern and Midwestern United States, and Ontario, Canada suffered a massive widespread power outage in August, 2003. A more recent example is the large blackout that occurred in Brazil in 2009, which plunged 40 % of the country into darkness. Since our daily routines would cease if these technological infrastructures were to disintegrate, maintaining the highest levels of availability in these networks is of crucial importance. Therefore, as a first step, we need to be able to assess the robustness of networks, which obviously depends on the type of disruption.

In this paper the performance indicator for robustness we will focus on is the availability, when the network is subject to probabilistic disruptions, rather than sabotage (see, for example, [2, 7]). The availability is defined as the fraction of time that a given system will be functioning as required [3]. For instance, for the traditional telephony service often a five nines (99.999 %) availability is guaranteed. The availability of a network generically depends on two factors: the availability of the individual network components and the topology of the network. The formula to compute the availability of a network component is $\mathrm{A} = \mathrm{MTBF}/(\mathrm{MTBF} + \mathrm{MTTR})$, where MTBF denotes the mean time to failure and MTTR the mean repair time. The topology of a network, of course, also effects its availability. For instance, higher redundancy in the network (e.g. more links connecting network nodes) will lead to a higher availability.

We will model a network subject to random component failures as a finite, undirected probabilistic graph $G(V, E)$ which consists of a set V of n nodes and collection E of m edges (all graphs in this paper are assumed to be undirected and finite, without self-loops). Nodes represent communication centers in the network while edges correspond to bidirectional communication links. We further assume that nodes are always operational while the edges are subjected to independent failures. More precisely we assume that each edge operates with link probability p independently. Note that the probability p can be interpreted as the availability of the links. The assumption that the nodes never fail is based upon the fact that node failures occur much less frequent than link failures, which occur, for example, when optical fibers are unintentionally broken by means of shovels.

We are interested in assessing the probability that any pair of nodes in the network can communicate with each other; equivalently, this is the probability that the corresponding graph G is connected. This is referred to as the *all-terminal reliability* (or simply the **reliability**) of the network. There is a long history of research into all-terminal reliability as a measure of robustness of the network, with most work directed at efficiently bounding the function [5].

The reliability polynomial $\mathrm{Rel}(G, p)$ is a polynomial in p indicating the probability that graph G contains at least one spanning tree, in other words, that the network is connected. The reliability polynomial is a function of both p and the topology of the graph G. The function $f = \mathrm{Rel}(G, p)$ is not identically 0 if we restrict to connected graphs G (which we do throughout), and the function then is increasing on $(0, 1)$, with $f(0) = 0$ and $f(1) = 1$. Moore and Shannon [9] proved that for every $C > 0$, every reliability polynomial crosses the curve

$$\frac{Cp}{1 - p(1 - C)}$$

at most once as p ranges from 0 to 1. In this paper we investigate the shape of the reliability curve, in terms of its concavity, and answer an open problem.

2 Points of inflection in (0, 1)

Colbourn [6] conjectured that reliability polynomials have at most one point of inflection in $(0, 1)$ (they need not have any points of inflection in the interval, as the graph consisting of a bundle of $k \geq 1$ edges has all-terminal reliability $1 - (1 - p)^k$, which has no point of inflection in $(0, 1)$).

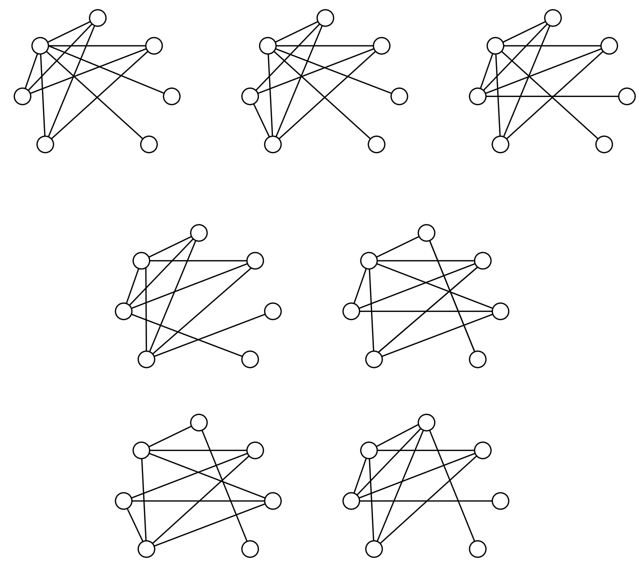

Fig. 1 All simple graphs of order at most 7 whose reliability functions have more than one point of inflection

Graves [8] considered points of inflection for general reliability systems, and showed that there are systems whose reliabilities have more than one point of inflection. However, these systems do not arise as all-terminal reliability of graphs. The issue of whether all-terminal reliability polynomials can have more than one point of inflection was settled recently [4], where an examination of the reliability of simple graphs (that is, those without multiple edges) of small order show that there are indeed graphs whose reliability polynomials have more than one point of inflection. Figure 1 shows the smallest simple graphs whose reliability polynomials have more than one point of inflection; all of these have 7 vertices and 10 edges, the same reliability function, namely $45p^6 - 128p^7 + 142p^8 - 72p^9 + 14p^{10}$ (see Fig. 2) and points of inflection at $p \approx 0.81898, 0.87866$. There are no other such simple graphs of order at most 7.

3 When do the reliability polynomials of graphs have at least one point of inflection?

We first turn to the question of when reliability polynomials have at least one point of inflection. We have seen that a bundle of k parallel edges has no point of inflection in $(0, 1)$, and the same is true for all trees. In fact, we can show that *every* connected graph G, with n vertices and m edges, can be embedded as an induced subgraph of a connected graph H such that the reliability polynomial of H has no point of inflection, and furthermore, we can take H to be simple if G is.

The argument is as follows. Let H_k be a graph formed from G by appending $k \geq 2$ leaves to vertices of G, in any

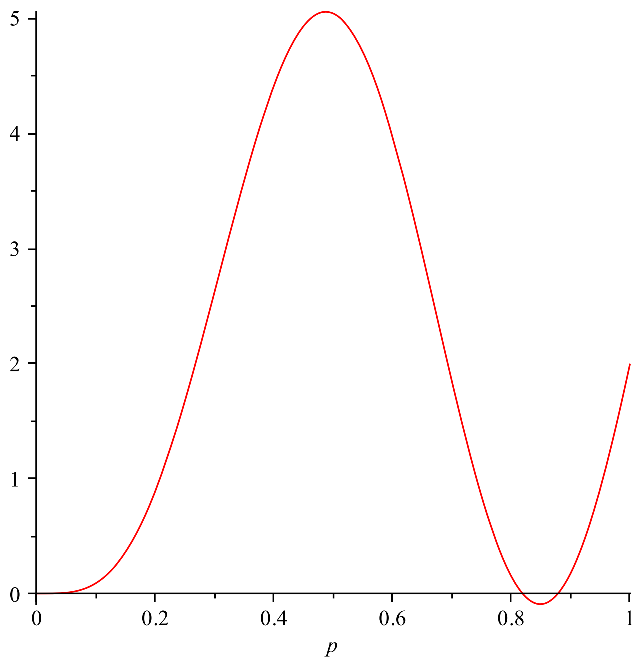

Fig. 2 Second derivative of the reliability polynomial of the graphs in Fig. 1

way. It is clear that $\mathrm{Rel}(H_k, p) = p^k \mathrm{Rel}(G, p)$. We write $\mathrm{Rel}(G, p)$ in its *F-form* (see, for example, [5]) as

$$\mathrm{Rel}(G, p) = \sum_{i=0}^{m-n+1} F_i (1-p)^i p^{m-i} \tag{1}$$

where F_i counts the number of subsets S of the edge set of G of size i whose removal leaves G connected. The F_i, arising as face numbers in a simplicial complex, satisfy *Sperner's Bound* [11],

$$(i+1)F_{i+1} \le (m-i)F_i \tag{2}$$

for all $i = 0, 1, \ldots, m-n$. A calculation shows that the second derivative of $\mathrm{Rel}(H_k, p) = p^k \sum F_i (1-p)^i p^{m-i}$ is equal to $p^{k-2} \sum L_i (1-p)^i p^{m-i}$ where

$$\begin{aligned} L_i &= k(k-1)F_i + 2k\big((m-i)F_i - (i+1)F_{i+1}\big) \\ &\quad + (i+2)(i+1)F_{i+2} - 2(i+1)(m-i-1)F_{i+1} \\ &\quad + (m-i)(m-i-1)F_i. \end{aligned} \tag{3}$$

Now for $i = 0, 1, \ldots, m-n+1$, we have that $F_i > 0$. Moreover, by Sperner's Bound (2), we have that $(m-i)F_i - (i+1)F_{i+1} \ge 0$. As the last three terms on the right side of (3) are constants with respect to k, it follows that for large enough k, $L_i > 0$ for $i = 0, 1, \ldots, m-n+1$, and so $p^{k-2} \sum L_i (1-p)^i p^{m-i} > 0$ on $(0, 1)$; that is, H_k, which has G as an induced subgraph (and is simple if G is) has no points of inflection in $(0, 1)$.

So how common are reliability polynomials with points of inflection? We can show that the reliability polynomials of almost all simple graphs (that is, almost all random graphs of order n in the Erdös–Rényi model $\mathcal{G}(n, \rho)$ with edge probability ρ) have a point of inflection in $(0, 1)$. To do so, we begin with a series of lemmas, each interesting in their own right. Note that for a connected graph G with its reliability's F-form given by (1), a calculation shows that $\mathrm{Rel}(G, p)''$ is equal to

$$\sum_{i=0}^{m-n+1} \big((i+2)(i+1)F_{i+2} - 2(m-i-1)(i+1)F_{i+1} + (m-i)(m-i-1)F_i\big)(1-p)^i p^{m-i-2}$$

where $F_j = 0$ for $j > m-n+1$. Recall that the *edge connectivity* of a graph is the minimum number of edges whose removal disconnects G (if G consists of a single vertex, we define its edge connectivity to be 0).

Proposition 1 *If G has edge connectivity $c \ge 2$, then the reliability of G is concave down near $p = 1$.*

Proof For $i < c$, from the definition of F_i we see that

$$F_i = \binom{m}{i},$$

and

$$F_c \le \binom{m}{c} - 1.$$

Thus for $i = 0, 1, \ldots, c-3$ we have

$$\begin{aligned} &(i+2)(i+1)F_{i+2} - 2(m-i-1)(i+1)F_{i+1} \\ &\quad + (m-i)(m-i-1)F_i \\ &= (i+2)(i+1)\binom{m}{i+2} - 2(m-i-1)(i+1)\binom{m}{i+1} \\ &\quad + (m-i)(m-i-1)\binom{m}{i} \\ &= \frac{m!}{(m-i-1)!i!} - 2\frac{m!}{(m-i-1)!i!} + \frac{m!}{(m-i-1)!i!} \\ &= 0, \end{aligned}$$

while for $i = c-2$, we have

$$\begin{aligned} &(i+2)(i+1)F_{i+2} - 2(m-i-1)(i+1)F_{i+1} \\ &\quad + (m-i)(m-i-1)F_i \\ &\le -c(c-1). \end{aligned}$$

It follows that for p sufficiently close to 1, $\mathrm{Rel}(G, p)'' \le -c(c-1) \le -2$, that is, $\mathrm{Rel}(G, p)$ is concave down near $p = 1$. □

Proposition 2 *Let G be a connected graph on at least* 3 *vertices. Then the reliability of G is concave up near* $p = 0$.

Proof In the F-form of the reliability of G (1), consider the largest j such that $F_j > 0$; clearly $j \le m - 2$ as G has at least 3 vertices. Then the dominant term of $\mathrm{Rel}(G, p)''$ when p is close to 0 is last term, $(m - j)(m - j - 1)F_j \times (1 - p)^j p^{m-j-2} > 0$, and hence $\mathrm{Rel}(G, p)$ is concave up near $p = 0$. □

We are now ready to prove the main result.

Theorem 1 *Let* $\rho \in (0, 1)$. *Then the reliability of almost every simple graph* $G \in \mathcal{G}(n, \rho)$ *of order n has at least one point of inflection*, *that is*, *the proportion of labeled simple graph on* $\{1, 2, \dots, n\}$ *whose reliability polynomial has at least one point of inflection tends to* 1 *as* $n \to \infty$.

Proof Almost every simple graph G has edge connectivity at least 2. To see this, note that if a graph G of order n has edge connectivity at most 1, then it has a partition of its vertices into two (nonempty) sets U and W with at most one edge between them; without loss, $|U| \le n/2$. If $E(U, W)$ is the event that there is at most one edge between U and W, then for $i = |U|$, we have

$$\mathrm{Prob}\big(E(U, W)\big) = (1 - \rho)^{i(n-i)} + i(n - i)\rho(1 - \rho)^{i(n-i)-1}.$$

It follows that the probability that G has edge connectivity less than 2 is at most

$$\sum_{i=1}^{\lfloor n/2\rfloor} \binom{n}{i}\big((1 - \rho)^{i(n-i)} + i(n - i)\rho(1 - \rho)^{i(n-i)-1}\big).$$

Now it is not hard to check that

$$\begin{aligned}\sum_{i=1}^{\lfloor n/2\rfloor} \binom{n}{i}(1 - \rho)^{i(n-i)} &\le \sum_{i=1}^{\lfloor n/2\rfloor} n^i(1 - \rho)^{in/2} \\ &\le \sum_{i=1}^{\lfloor n/2\rfloor} \big(n(1 - \rho)^{n/2}\big)^i \\ &< \sum_{i=1}^{\infty} \big(n(1 - \rho)^{n/2}\big)^i \\ &= n(1 - \rho)^{n/2}\frac{1}{1 - n(1 - \rho)^{n/2}} \\ &< 2n(1 - \rho)^{n/2}\end{aligned}$$

for n sufficiently large. It follows from this that the probability that G does not have edge connectivity at least 2 is less than

$$2n(1 - \rho)^{n/2} + 2n^3\rho(1 - \rho)^{-1+n/2}$$

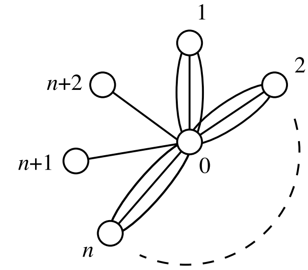

Fig. 3 Graph G_n of order $n + 3$

which tends to 0 as $n \to \infty$.

For any such graph, from Propositions 2 and 1 its reliability changes from concave up to concave down over $(0, 1)$, and hence it has a point of inflection in the interval. □

4 Infinite families of graph whose reliability polynomials have more than one point of inflection

We can show that there are indeed infinitely many graphs that have more than one point of inflection.

Theorem 2 *For* $n \ge 1$ *let* G_n *be the graph on vertex set* $\{0, 1, \dots, n + 2\}$ *formed from the star* $K_{1,n+2}$ *by replacing n of the edges by a set of three edges in parallel* (*see Fig.* 3). *Then* $\mathrm{Rel}(G_n, p)$ *has at least two points of inflection for any* $n \ge 5$.

Proof The reliability of G_n is given by

$$f_n(p) = p^2\big(1 - (1 - p)^3\big)^n.$$

To simplify matters, we make the substitution $q = 1 - p$, and observe that $q_0 \in (0, 1)$ is a point of inflection of

$$g_n(q) = (1 - q)^2\big(1 - q^3\big)^n$$

if and only if $p_0 = 1 - q_0$ is a point of inflection of $f_n(p)$. Thus it suffices to show that $g = g_n(q)$ has two points of inflection in $(0, 1)$.

With a bit of calculation, we find that

$$\begin{aligned}\frac{g''}{(1 - q)^2(1 - q^3)^{n-2}} &= \big(9n^2 + 9n + 2\big)q^4 + (12n + 4)q^3 \\ &\quad + (12n + 6)q^2 - (6n - 4)q + 2.\end{aligned}$$

We set

$$\begin{aligned}h_n(q) = {}&\big(9n^2 + 9n + 2\big)q^4 + (12n + 4)q^3 + (12n + 6)q^2 \\ &- (6n - 4)q + 2.\end{aligned}$$

Note that close to 0 and close to 1, $h = h_n(q)$ is positive. If we can show that $h_n(q)$ is negative somewhere between 0 and 1, then $h_n(q)$ (and hence $g'' = (1-q^2)(1-q^3)^{n-2}h$) will change sign twice in (0, 1). This will show that $g_n(q)$, and therefore $f_n(p)$, the reliability of G_k, has at least two points of inflection in (0, 1).

A visual inspection of some plots of h_n reveals that indeed h_n takes on negative values in (0, 1), where the location of the roots of h_n seems to move to the left for large n. In fact, evaluating h_n at $q = 1/n$, we find that

$$h_n(1/n) = -4 + \frac{16}{n} + \frac{27}{n^2} + \frac{13}{n^3} + \frac{2}{n^4},$$

which is clearly a decreasing function of n. Moreover, $h_6(1/6) = -169/324 < 0$, so that $h_n(1/n)$ is negative for $n \geq 6$. It follows that h changes sign (at least) twice in (0, 1) for $n \geq 6$. Direct computation shows that this holds as well for $n = 5$, so we conclude that $f_n(p)$, the reliability of G_n, has at least two points of inflection in (0, 1) for all $n \geq 5$. □

One can see that $h_n(q)$ has, in fact, exactly two sign changes for any $n \geq 1$. Descartes' Rule of Signs (see, for example, [1, p. 171]) states that the number of positive roots of a polynomial with real coefficients is the equal to the number of sign changes between nonzero coefficients, or less than that by a multiple of 2. It follows that $h_n(q)$ has at most two positive roots, and hence $\mathrm{Rel}(G_n, p)$ has *exactly* two points of inflection in (0, 1) for any $n \geq 5$.

We also remark that the reliability of a graph formed from *any* tree of order n, by replacing each edge by a set of three edges in parallel, and then adding two leaves, will have the same reliability as the graph G_n, and hence will provide as well examples of graphs whose reliability polynomial has exactly two points of inflection in (0, 1).

What about points of inflection for the reliability of simple graphs? Are the multiple edges necessary for the reliability to have more than one point of inflection? The answer is, in fact, no. From numerical computations we have verified that for $n \leq 100$, the graphs formed by adding two leaves to $K_{3,n}$, the complete bipartite graph with cells of size 3 and n (also known as *triple stars* [12]), leads to examples of simple graphs with two inflection points. The next theorem shows that these are indeed examples of simple graphs whose reliability polynomials have more than one point of inflection, provided n is large enough.

Theorem 3 *Let H_n be the graph formed from the complete bipartite graph $K_{3,n}$ by attaching two new vertices of degree* 1 *each to a vertex of $K_{3,n}$. Then* $\mathrm{Rel}(H_n, p)$ *has at least two points of inflection for n sufficiently large.*

Proof The reliability of $K_{3,n}$ is known to be (see [12])

$$\mathrm{Rel}(K_{3,n}, p) = p^n\big((3-3p+p^2)^n - 3(1-p)^n(3-2p)^n + 2\cdot 3^n(1-p)^{2n}\big),$$

so it follows that

$$\mathrm{Rel}(H_n, p) = h_n(p) = p^{n+2}\big((3-3p+p^2)^n - 3(1-p)^n(3-2p)^n + 2\cdot 3^n(1-p)^{2n}\big).$$

By Proposition 2 of the previous section, $\mathrm{Rel}(H_n, p)$ is concave up at $p = 0$. A computation shows that for $n \geq 3$, $\mathrm{Rel}(H_n, p)$ is concave up at $p = 1$ because $h_n''(1) = 2$.

It remains to show that for some $p \in (0, 1)$, the second derivative of $\mathrm{Rel}(H_n, p)$ is negative, or equivalently, that $h_n''(p) < 0$ for some $p \in (0, 1)$. Another calculation shows that

$$\lim_{n\to\infty} h_n''(1 - 1/n) = -4e < 0.$$

It follows that for n sufficiently large, the second derivative of $\mathrm{Rel}(H_n, p)$ is negative at $p = 1 - 1/n$, and H_n has at least two points of inflection in (0, 1). □

We remark that $K_{3,n}$ was suggested by [10] as a survivable optical network structure.

The question whether any graph, simple or otherwise, can have more than two points of inflection, remains open. We pose a final question: what is the closure of the inflection points in (0, 1) of reliability polynomials? Figure 4 plots the points of inflection of all simple graphs of order 9, and we

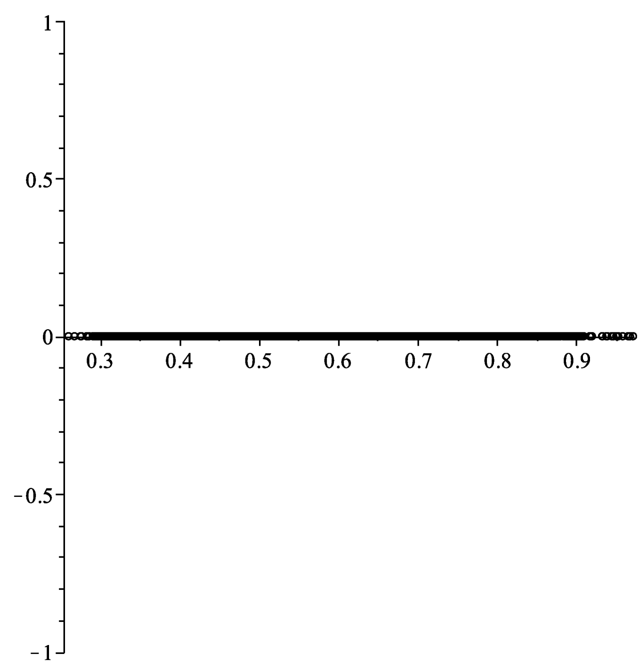

Fig. 4 Inflection points in (0, 1) of the reliability polynomials of simple graphs of order 9

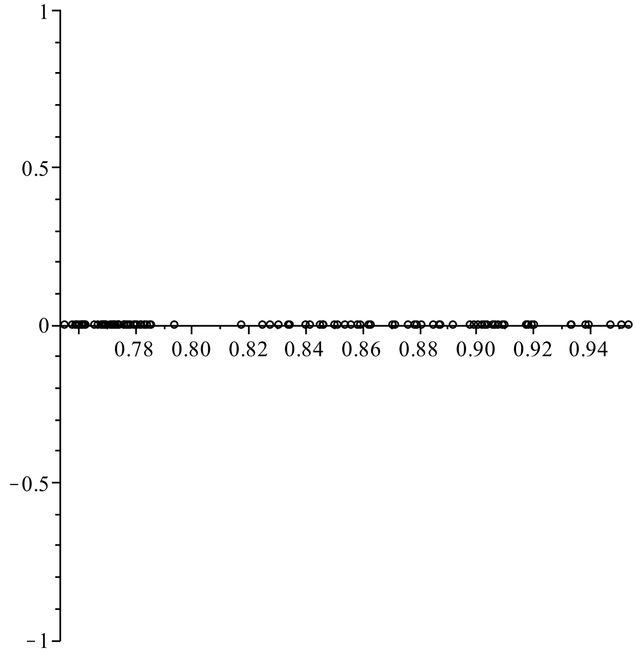

Fig. 5 Inflection points in (0, 1) of the reliability polynomials of simple graphs of order 9 with multiple points of inflection in the interval

are led to conjecture that the closure of the inflection points in (0, 1) is, in fact [0, 1]. When looking at the points of inflections in (0, 1) for reliability polynomials with more than one point of inflection in the interval, we, of course, get a sparser picture (see Fig. 5), and it may be true that there is a inflection point free open interval in [0, 1].

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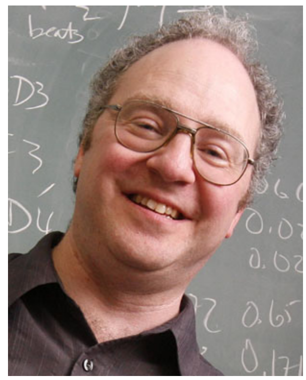

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